Exercise Sheet 3

Exercise 1 (Regular Subgroups are closed)**.** Let G be a Lie group, $H \leq G$ a subgroup that is also a regular submanifold. Prove that H is a closed subgroup of G.

Exercise 2 (Non-closed Subgroup)**.** Give an example of a Lie group G and a subgroup $H < G$ that is not closed and not a Lie group with the topology induced from G.

Exercise 3 (Differential of det)**.** We consider the determinant function $\det : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^*$. Show that its differential at the identity matrix I is the trace function

$$D_I \det = \mathrm{tr}.$$

Exercise 4 (The Matrix Lie Groups $O(p, q)$ and $U(p, q)$)**.** Let $p, q \in \mathbb{N}$ and $n = p + q$.

a) We define the (indefinite) symmetric bilinear form $\langle \cdot, \cdot \rangle_{p,q}$ of signature (p, q) on $\mathbb{R}^n$ to be

$$\langle v, w \rangle_{p,q} := v_1 w_1 + \cdots + v_p w_p - v_{p+1} w_{p+1} - \cdots - v_{p+q} w_{p+q}$$

for all $v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_n) \in \mathbb{R}^n$. As the orthogonal group $O(n)$ is defined to be the group of matrices that preserve the standard Euclidean inner product we may now define $O(p, q)$ to be the group of matrices that preserve the above bilinear form:

$$O(p, q) := \{A \in \mathrm{GL}(n, \mathbb{R}) : \langle Av, Aw \rangle_{p,q} = \langle v, w \rangle_{p,q} \quad \forall v, w \in \mathbb{R}^n\}.$$

Show that $O(p, q)$ is a Lie group using the inverse function theorem/constant rank theorem. What is its dimension?

b) Similarly we may define the following symmetric sesquilinear form on $\mathbb{C}^n$

$$\langle w, z \rangle_{p,q} := \bar{w}_1 z_1 + \cdots + \bar{w}_p z_p - \bar{w}_{p+1} z_{p+1} - \cdots - \bar{w}_{p+q} z_{p+q}$$

for all $w = (w_1, \ldots, w_n), z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, and

$$U(p, q) = \{A \in \mathrm{GL}(n, \mathbb{C}) : \langle Aw, Az \rangle_{p,q} = \langle w, z \rangle_{p,q} \quad \forall w, z \in \mathbb{C}^n\}.$$

Show that $U(p, q)$ is a (real) Lie group using the inverse function theorem/constant rank theorem. What is its (real) dimension?

Exercise 5 (Dimension of $\mathrm{O}(n, \mathbb{R})$)**.** Show that the dimension of $\mathrm{O}(n, \mathbb{R})$ is $n(n-1)/2$.

Exercise 6 (One- and two-dimensional Lie Algebras)**.** Classify the one- and two-dimensional real Lie algebras up to Lie algebra isomorphism and realize them as Lie subalgebras of some $\mathfrak{gl}_n \mathbb{R} = \mathfrak{gl}(\mathbb{R}^n)$.

Hint: In dimension two one can show that if the Lie algebra is non-abelian then there is a basis X, Y such that $[X, Y] = Y$.

Exercise 7 (The adjoint representation ad)**.** Let V be a vector space over a field k.

a) Show that the vector space of endomorphisms

$$\mathfrak{gl}(V) \coloneqq \{A\colon V \to V \text{ linear}\}$$

is a Lie algebra with the Lie bracket given by the commutator

$$[A, B] \coloneqq AB - BA$$

for all $A, B \in \mathfrak{gl}(V)$.

b) Let $\mathfrak{g}$ be a Lie algebra over k. The *adjoint representation*

$$\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

is defined as $\mathrm{ad}(X)(Y) \coloneqq [X, Y]$ for all $X, Y \in \mathfrak{g}$. Show that ad is a Lie algebra homomorphism.

Exercise 8 (Quaternions)**.** Let $\mathbb{H} := \mathbb{R} + \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ and define – in addition to the $\mathbb{R}$ vector space structure – a multiplication on $\mathbb{H}$ by requiring:

$$\begin{aligned} \mathbf{ij} &= \mathbf{k} = -\mathbf{ji}, \\ \mathbf{jk} &= \mathbf{i} = -\mathbf{kj}, \\ \mathbf{ki} &= \mathbf{j} = -\mathbf{ik}, \\ \mathbf{i}^2 &= \mathbf{j}^2 = \mathbf{k}^2 = -1. \end{aligned}$$

The resulting skew-field is called the Hamiltonian quaternions.

a) Prove that there is a ring isomorphism:

$$\mathbb{H} \cong \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \middle| a, b \in \mathbb{C} \right\}.$$

b) Define a Lie bracket on $\mathbb{H}$ by $[u, v] := uv - vu$. Show that $V = \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ is a Lie ideal in $\mathbb{H}$ and that the Lie subalgebra $(V, [\cdot, \cdot])$ is isomorphic to the Lie algebra $\mathbb{R}^3$ with the cross product

$$x \times y = (x_2 y_3 - y_2 x_3, x_3 y_1 - y_3 x_1, x_1 y_2 - y_1 x_2) \quad \forall x, y \in \mathbb{R}^3$$

as a Lie bracket.

Remark: A *Lie ideal* in a Lie algebra $\mathfrak{g}$ is a Lie subalgebra $\mathfrak{i} \subseteq \mathfrak{g}$ such that $[X, Y] \in \mathfrak{i}$ for all $X \in \mathfrak{g}, Y \in \mathfrak{i}$.