Exercise Sheet 5

Exercise 1 (Discrete Subgroups of $\mathbb{R}^n$). Let $D < \mathbb{R}^n$ be a discrete subgroup. Show that there are $x_1, \dots, x_k \in D$ such that

- $x_1, \dots, x_k$ are linearly independent over $\mathbb{R}$, and
- $D = \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_k$, i.e. $x_1, \dots, x_k$ generate D as a $\mathbb{Z}$-submodule of $\mathbb{R}^n$.

Exercise 2 (Surjectivity of the matrix exponential).

a) Show that the exponential map of $\mathrm{GL}(2, \mathbb{C})$ is surjective.

b) Show that the exponential map of $\mathrm{U}(n)$ is surjective.

c) Show that the exponential map of $\mathrm{SO}(2, \mathbb{R})$ is surjective.

Exercise 3 (Abstract Subgroups as Lie Subgroups). Let H be an abstract subgroup of a Lie group G and let $\mathfrak{h}$ be a subspace of the Lie algebra $\mathfrak{g}$ of G. Further let $U \subseteq \mathfrak{g}$ be an open neighborhood of $0 \in \mathfrak{g}$ and let $V \subseteq G$ be an open neighborhood of $e \in G$ such that the exponential map $\exp : U \to V$ is a diffeomorphism satisfying $\exp(U \cap \mathfrak{h}) = V \cap H$. Show that the following statements hold:

a) H is a Lie subgroup of G with the induced relative topology;

b) $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$;

c) $\mathfrak{h}$ is the Lie algebra of H.

Exercise 4 (Lie Group homomorphisms and their differentials). Let G be a connected Lie group, let H be a Lie group and let $\varphi, \psi \colon G \to H$ be Lie group homomorphisms.

Show that $\varphi = \psi$ if and only if $d\varphi = d\psi$.

Exercise 5 (Multiplication and exp). Let G be a Lie group with Lie algebra $\mathfrak{g}$. Show that for all $X, Y \in \mathfrak{g}$ and small enough $t \in \mathbb{R}$

$$\exp(tX)\exp(tY) = \exp(t(X+Y) + O(t^2))$$

where $O(t^2)$ is a differentiable $\mathfrak{g}$-valued function such that $\frac{O(t^2)}{t^2}$ is bounded as $t \to 0$.