Exercise Sheet 6

Exercise 1 (Isomorphism theorems for Lie algebras)**.** Let $\mathfrak{g}$ be a Lie algebra.

a) Let $\mathfrak{h} \trianglelefteq \mathfrak{g}$ be an ideal. Show that

$$[X + \mathfrak{h}, Y + \mathfrak{h}] := [X, Y] + \mathfrak{h}$$

defines a Lie algebra structure on $\mathfrak{g}/\mathfrak{h}$.

b) Show that if $\varphi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism then

$$\mathfrak{g}/\ker\varphi \cong \mathrm{im}\varphi$$

as Lie algebras.

c) Let $\mathfrak{h} \subseteq \mathfrak{I}$ be ideals of $\mathfrak{g}$. Show that

$$\mathfrak{I}/\mathfrak{h} \trianglelefteq \mathfrak{g}/\mathfrak{h} \quad \text{and} \quad (\mathfrak{g}/\mathfrak{h})/(\mathfrak{I}/\mathfrak{h}) \cong \mathfrak{g}/\mathfrak{I}.$$

d) Let $\mathfrak{h}$ and $\mathfrak{I}$ be ideals of $\mathfrak{g}$. Show that $\mathfrak{h} + \mathfrak{I}$ and $\mathfrak{h} \cap \mathfrak{I}$ are ideals in $\mathfrak{g}$, and that

$$\mathfrak{h}/(\mathfrak{h} \cap \mathfrak{I}) \cong (\mathfrak{h} + \mathfrak{I})/\mathfrak{I}.$$

Exercise 2 (Solvable Lie algebras)**.** a) Show that Lie subalgebras and homomorphic images of solvable Lie algebras are solvable.

b) Show that if $\mathfrak{h}$ and $\mathfrak{I}$ are solvable ideals of a Lie algebra $\mathfrak{g}$ then $\mathfrak{h} + \mathfrak{I}$ is a solvable ideal.

Hint: Use exercise 1. d).

c) Deduce that every Lie algebra contains a unique maximal solvable ideal.

Exercise 3 (Quotients of Lie groups)**.** Let G be a Lie group and let $K \leq G$ be a closed normal subgroup.

Show that G/K can be equipped with a Lie group structure such that the quotient map $\pi\colon G \to G/K$ is a surjective Lie group homomorphism with kernel K.

Exercise 4 (Common eigenvectors)**.** Let G be a connected Lie group and let $\pi\colon G \to \mathrm{GL}(V)$ be a finite-dimensional complex representation.

A *common eigenvector of* $\{\pi(g) : g \in G\}$ is a vector $v \in V$ such that there is a smooth homomorphism $\chi\colon G \to \mathbb{C}$ with $\pi(g)v = \chi(g) \cdot v$ for all $g \in G$. Similarly, a *common eigenvector of* $\{d_e\pi(X)\colon X \in \mathfrak{g}\}$ is a vector $v \in V$ such that there is a linear functional $\lambda\colon \mathfrak{g} \to \mathbb{C}$ with $d_e\pi(X)v = \lambda(X) \cdot v$ for all $X \in \mathfrak{g}$.

Show that a vector $v \in V$ is a common eigenvector of $\{d_e\pi(X) : X \in \mathfrak{g}\}$ if and only if it is a common eigenvector of $\{\pi(g) : g \in G\}$. Moreover, show that $\chi(\exp(X)) = e^{\lambda(X)}$ for all $X \in \mathfrak{g}$ (with $\chi\colon G \to \mathbb{C}$ and $\lambda\colon \mathfrak{g} \to \mathbb{C}$ as above).

Exercise 5 (Weight spaces and ideals)**.** Let $\mathfrak{g}$ be a Lie algebra, let $\mathfrak{h} \trianglelefteq \mathfrak{g}$ be an ideal and let $\pi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ a finite-dimensional complex representation. For a given linear functional $\lambda\colon \mathfrak{h} \to \mathbb{C}$ consider its weight space

$$V_\lambda^{\mathfrak{h}} := \{v \in V \mid \pi(X)v = \lambda(X)v \quad \forall X \in \mathfrak{h}\}.$$

Show that every weight space $V_\lambda^{\mathfrak{h}}$ is invariant under $\pi(\mathfrak{g})$, i.e. $\pi(Y)V_\lambda^{\mathfrak{h}} \subseteq V_\lambda^{\mathfrak{h}}$ for every $\lambda \in \mathfrak{h}^*, Y \in \mathfrak{g}$.

Exercise 6 (Lie's theorem for Lie algebras)**.** Let $\mathfrak{g}$ be a solvable Lie algebra and let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional complex representation.

Show that $\rho(\mathfrak{g})$ stabilizes a flag $V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_n = 0$, with $\mathrm{codim} V_i = i$, i.e. $\rho(X)V_i \subseteq V_i$ for every $X \in V_i$, $i = 1, \ldots, n$.

<u>Hint:</u> Use exercise 5.