Solutions Exercise Sheet 2

Exercise 1 (Transitive Group Actions)**.** Let G be a topological group, X a topological space and $\mu : G \times X \to X$ a continuous transitive group action, i.e. for any two $x, y \in X$ there is $g \in G$ such that $\mu(g, x) = g \cdot x = y$.

a) Show that if G is compact then X is compact.

b) Show that if G is connected then X is connected.

Solution. Let $x_0 \in X$ and consider the map

$$\begin{aligned} \varphi : G &\to X, \\ g &\mapsto \mu(g, x_0). \end{aligned}$$

Because μ is a continuous action the map φ is continuous too. Further the action μ is transitive, i.e. for every $y \in X$ there is a $g \in G$ such that $\mu(g, x_0) = y$. In other words, φ is surjective.

Part a) follows from the fact that $X = \varphi(G)$ is compact as the image of a compact group.

Part b) follows from the fact that continuous maps send connected components to connected components and again that $\varphi(G) = X$.

Exercise 2 (Examples of Haar Measures)**.** We start with a general remark about the regularity of the measures in the exercise.

Theorem (Thm 7.8 in Folland, Real Analysis: Modern Techniques and Their Applications)**.** Let X be locally compact second countable Hausdorff space. Then every Borel measure on X that is finite on compact sets is regular.

The measures we consider in this exercise are defined on subspaces X of $\mathbb{R}^k$ for some $k \in \mathbb{N}$, which are equipped with the subspace topology. In particular, if $K \subset X$ is compact, then it is compact also in $\mathbb{R}^k$.
Moreover, these measures (with the exception of part d)) are of the form $f(x)\mathrm{d}\mathcal{L}(x)$, where $d\mathcal{L}$ denotes the Lebesgue measure and f is some continuous function on X. Thus they are finite on compact sets and by the above theorem they are regular.

a) Let us consider the *three-dimensional Heisenberg group* $H = \mathbb{R} \rtimes_\eta \mathbb{R}^2$, where $\eta : \mathbb{R} \to \mathrm{Aut}(\mathbb{R}^2)$ is defined by

$$\eta(x) \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} y \\ z + xy \end{pmatrix},$$

for all $x, y, z \in \mathbb{R}$. Thus the group operation is given by

$$(x_1, y_1, z_1) * (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2)$$

and it is easy to see that it can be identified with the matrix group

$$H \cong \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

Verify that the Lebesgue measure is the Haar measure of $\mathbb{R} \rtimes_\eta \mathbb{R}^2$ and that the group is unimodular.

Solution. Denote by μ the measure on H induced by the Lebesgue measure on $\mathbb{R}^3$. In order to show that μ is unimodular we need to see that

$$\mu(\lambda(h)f) = \mu(f) = \mu(\rho(h)f)$$

for every $f \in C_c(H)$, $h \in H$.

Let $h_1 = (x_1, y_1, z_1) \in H$ and $f \in C_c(H)$. We compute

$$\begin{aligned}
&\int (\lambda(h_1^{-1})f)(x_2, y_2, z_2) dx_2 dy_2 dz_2 \\
&= \int f(x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2) dx_2 dy_2 dz_2 \\
&\overset{\text{Fubini}}{=} \int f(x_1 + x_2, y_1 + y_2, z_2 + (z_1 + x_1 y_2)) dz_2 dx_2 dy_2 \\
&\overset{\text{transl. inv.}}{=} \int f(x_1 + x_2, y_1 + y_2, z_2) dz_2 dx_2 dy_2 \\
&\overset{\text{F. \& t.i.}}{=} \int f(x_1, y_1 + y_2, z_2) dx_2 dy_2 dz_2 \\
&\overset{\text{F. \& t.i.}}{=} \int f(x_1, y_2, z_2) dx_2 dy_2 dz_2.
\end{aligned}$$

This shows left-invariance.

$$
\begin{aligned}
&\int (\rho(h_1)f)(x_2,y_2,z_2)dx_2dy_2dz_2 \\
&= \int f(x_2+x_1, y_2+y_1, z_2+z_1+x_2y_1)dx_2dy_2dz_2 \\
&\overset{\text{Fubini}}{=} \int f(x_1+x_2, y_1+y_2, z_2+(z_1+x_2y_1))dz_2dx_2dy_2 \\
&\overset{\text{transl. inv.}}{=} \int f(x_1+x_2, y_1+y_2, z_2)dz_2dx_2dy_2 \\
&\overset{\text{F. \& t.i.}}{=} \int f(x_1, y_1+y_2, z_2)dx_2dy_2dz_2 \\
&\overset{\text{F. \& t.i.}}{=} \int f(x_1, y_2, z_2)dx_2dy_2dz_2.
\end{aligned}
$$

This shows right-invariance. Therefore μ is a left- and right-invariant Haar measure on H and H is unimodular.

b) Let

$$
P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in \mathbb{R}, a \neq 0 \right\}.
$$

Show that $\frac{da}{a^2}\, db$ is the left Haar measure and $da\, db$ is the right Haar measure. In particular, P is *not* unimodular.

Solution. Let $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in P$ and $f \in C_c(P)$. We compute

$$
\begin{aligned}
&\int \left(\lambda \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}^{-1} \right) f \right) \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \frac{dx}{x^2} dy \\
&= \int f \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \right) \frac{dx}{x^2} dy \\
&= \int f \begin{pmatrix} ax & ay + bx^{-1} \\ 0 & a^{-1}x^{-1} \end{pmatrix} a^2 \frac{dx}{(ax)^2} dy = \dots
\end{aligned}
$$

we change coordinates to $\bar{x} = ax, \bar{y} = ay$ which has Jacobi determinant a^2

$$
\begin{aligned}
\dots &= \int f \begin{pmatrix} \bar{x} & \bar{y} + ab\bar{x}^{-1} \\ 0 & \bar{x}^{-1} \end{pmatrix} \frac{d\bar{x}}{\bar{x}^2} d\bar{y} \\
&= \int f \begin{pmatrix} \bar{x} & \bar{y} + ab\bar{x}^{-1} \\ 0 & \bar{x}^{-1} \end{pmatrix} d\bar{y} \frac{d\bar{x}}{\bar{x}^2} \\
&= \int f \begin{pmatrix} \bar{x} & \bar{y} \\ 0 & \bar{x}^{-1} \end{pmatrix} \frac{d\bar{x}}{\bar{x}^2} d\bar{y}.
\end{aligned}
$$

This shows left-invariance for the measure $\frac{dx}{x^2}\,dy$ as claimed.
We will now see that $da\,db$ is right-invariant:

$$\begin{aligned}
&\int \left(\rho\left(\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}\right) f\right)\begin{pmatrix} x & y \\ 0 & x^{-1}\end{pmatrix} dxdy \\
&= \int f\left(\begin{pmatrix} x & y \\ 0 & x^{-1}\end{pmatrix}\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}\right) dxdy \\
&= \int f\left(\begin{pmatrix} ax & bx + a^{-1}y \\ 0 & a^{-1}x^{-1}\end{pmatrix}\right) dxdy = \dots
\end{aligned}$$

we change coordinates to $\bar{x} = ax, \bar{y} = a^{-1}y$ which has Jacobi determinant 1

$$\begin{aligned}
\dots &= \int f\left(\begin{pmatrix} \bar{x} & ba^{-1}\bar{x} + \bar{y} \\ 0 & \bar{x}^{-1}\end{pmatrix}\right) d\bar{x}d\bar{y} \\
&\overset{\text{F \& t.i}}{=} \int f\left(\begin{pmatrix} \bar{x} & \bar{y} \\ 0 & \bar{x}^{-1}\end{pmatrix}\right) d\bar{x}d\bar{y}
\end{aligned}$$

This shows right-invariance. Since both measures clearly do not coincide P is *not* unimodular.

c) Let $G := \mathrm{GL}_n(\mathbb{R}) \subseteq \mathbb{R}^{n^2}$ denote the group of invertible matrices over $\mathbb{R}$. Let λ_{n^2} denote the Lebesgue measure on $\mathbb{R}^{n^2}$. Prove that

$$\mathrm{d}m(x) := |\det x|^{-n}\,\mathrm{d}\lambda_{n^2}(x)$$

defines a bi-invariant (i.e. left- and right-invariant) Haar measure on G.

Solution. As $\mathrm{GL}_n(\mathbb{R}) = \det^{-1}\big(\mathbb{R}\backslash\{0\}\big)$ is open in $\mathbb{R}^{n^2}$, $\lambda_{n^2}\big|_{\mathrm{GL}_n(\mathbb{R})}$ assigns non-zero measure to non-empty open and finite measure to compact subsets of $\mathrm{GL}_n(\mathbb{R})$ (if $K \subseteq \mathrm{GL}_n(\mathbb{R})$ is compact in $\mathrm{GL}_n(\mathbb{R})$ and $\mathcal{U}$ an open cover of K in $\mathbb{R}^{n^2}$, then $\mathcal{U} \cap \mathrm{GL}_n(\mathbb{R}) := \{U \cap \mathrm{GL}_n(\mathbb{R}); U \in \mathcal{U}\}$ is an open cover of K in $\mathrm{GL}_n(\mathbb{R})$, thus it admits a finite subcover and hence so does $\mathcal{U}$). As det is continuous and does not vanish on $\mathrm{GL}_n(\mathbb{R})$, the above also holds for $dm(g) := |\det g|^{-n}\, d\lambda_{n^2}(g)$.
It remains to show that m is invariant. To this end we note that for $g \in \mathrm{GL}_n(\mathbb{R})$, if $g = (g_1, \dots, g_n)$ and $h \in \mathrm{GL}_n(\mathbb{R})$, then

$$hg = (hg_1, \dots, hg_2) \quad (g \in \mathrm{Mat}_n(\mathbb{R})),$$

so that the left-action of h on $\mathrm{GL}_n(\mathbb{R})$ can be viewed as a restriction of a diagonal matrix $\mathrm{diag}(h, \dots, h) \in \mathbb{R}^{n^2 \times n^2}$ acting on a subset of $\mathbb{R}^{n^2}$. This means that for $F : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R}), g \mapsto F(g) \coloneqq hg$ it holds

$$\det DF(g) = (\det h)^n.$$

Let $f \in C_c\big(\mathrm{GL}_n(\mathbb{R})\big)$, then

$$\begin{aligned}
\int_{\mathrm{GL}_n(\mathbb{R})} f(hg)\,|\mathrm{det}g|^{-n}\; d\lambda_{n^2}(g) &= \int_{\mathrm{GL}_n(\mathbb{R})} f(hg)\,|\mathrm{det}hg|^{-n}\,|\mathrm{det}h|^{n}\; d\lambda_{n^2}(g) \\
\big(\varphi(x) = f(x)\,|\mathrm{det}x|^{-n}\big) \quad &= \int_{\mathrm{GL}_n(\mathbb{R})} \varphi(F(g))\,|\mathrm{det}DF(g)|^{n}\; d\lambda_{n^2}(g) \\
(\text{change of variables}) \quad &= \int_{F(\mathrm{GL}_n(\mathbb{R}))} \varphi(y) d\lambda_{n^2}(y) \\
&= \int_{h\cdot \mathrm{GL}_n(\mathbb{R})} f(y)\,|\mathrm{det}y|^{-n}\; d\lambda_{n^2}(y) \\
&= \int_{\mathrm{GL}_n(\mathbb{R})} f(y)\,|\mathrm{det}y|^{-n}\; d\lambda_{n^2}(y).
\end{aligned}$$

This proves that m is a left Haar measure on $\mathrm{GL}_n(\mathbb{R})$. The measure is also right-invariant, because the map

$$g \mapsto \begin{pmatrix} g_1 h \\ \vdots \\ g_n h \end{pmatrix}$$

does also have Jacobian $|\mathrm{det}h|^n$ (for example because $gh = (h^t g^t)^t$ and the Jacobian of transposition – being an idempotent map – is equal to 1). Thus $\mathrm{GL}_n(\mathbb{R})$ is unimodular.

d) Let $G = \mathrm{SL}_n(\mathbb{R})$ denote the group of matrices of determinant 1 in $\mathbb{R}^{n\times n}$. For a Borel subset $B \subseteq \mathrm{SL}_n(\mathbb{R})$ define

$$m(B) := \lambda_{n^2}\big(\{tg; g \in B, t \in [0,1]\}\big).$$

Show that m is a well-defined bi-invariant Haar measure on $\mathrm{SL}_n(\mathbb{R})$.

Solution. To check well-definedness we have to check that for any Borel subset $B \subseteq \mathrm{SL}_n(\mathbb{R})$ the cone

$$\mathcal{C}(B) = \{tb : b \in B, t \in [0,1]\}$$

is a Borel subset of $\mathbb{R}^{n^2}$. To this end we note first that

$$\mathcal{C}(B) = \mathcal{C}'(B) \cup \{0\},$$

where

$$\mathcal{C}'(B) = \{tb : b \in B, t \in (0,1]\}.$$

It clearly suffices to show that $\mathcal{C}'(B)$ is Borel. To this end let

$$\mathrm{GL}_n^{\pm 1}(\mathbb{R}) = \{g \in \mathrm{GL}_n(\mathbb{R}); |\mathrm{det}g| = 1\}.$$

Note that $\mathrm{GL}_n^{\pm 1}(\mathbb{R})$ is homeomorphic to a disjoint union of two copies of $\mathrm{SL}_n(\mathbb{R})$, in particular B is Borel in $\mathrm{GL}_n^{\pm 1}(\mathbb{R})$. (As groups $\mathrm{GL}_n^{\pm 1}(\mathbb{R}) \cong \mathrm{SL}_n(\mathbb{R}) \rtimes C_2$, where C_2 is the group with two elements.) Define

$$\Psi : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n^{\pm 1}(\mathbb{R}), \quad g \mapsto \frac{1}{\sqrt[n]{|\mathrm{det}g|}}\, g.$$

This is a Borel map and therefore

$$\mathcal{C}'(B) = \Psi^{-1}(B) \cap \det{}^{-1}(0,1]$$

is measurable.

- $\subseteq$ Let $t \in (0,1]$, and $b \in B$. Then $x = tb$ satisfies $\det(x) = t^n \det(b) = t^n \in (0,1]$ and $\Psi(x) = \Psi(tb) = \frac{tb}{\sqrt[n]{t^n}} = b \in B$. Thus $tb \in \Psi^{-1}(B) \cap \det^{-1}(0,1]$.
- $\supseteq$ Let $x \in \Psi^{-1}(B)$ with $\det(x) \in (0,1]$ and let $b \in B$ be such that $\Psi(x) = \frac{x}{\sqrt[n]{|\det x|}} = b$. Then $x = \sqrt[n]{|\det x|}b = tb$ with $t = \sqrt[n]{|\det x|} \in (0,1]$.

Thus we have $\lambda_{n^2}(\mathcal{C}'(B))$ is well-defined and we only have to check that $m(B) = \lambda_{n^2}(\mathcal{C}'(B))$ defines a measure which is finite on compact sets. But this follows directly from the fact that $B \mapsto \mathcal{C}'(B)$ preserves intersections, unions, disjoint unions and compact sets.

The final claim now follows immediately from the argument in part c), which realizes the action of an element $g \in \mathrm{SL}_n(\mathbb{R})$ on $\mathbb{R}^{n^2}$ as a diagonal action of n copies of g, together with the fact that $\Phi_* \lambda_{n^2} = |\det \Phi| \, \lambda_{n^2}$ for linear Φ, $\det g = 1$, $\mathcal{C}(gB) = g\mathcal{C}(B)$ and $\mathcal{C}(Bg) = \mathcal{C}(B)g$ for all $g \in \mathrm{SL}_n(\mathbb{R})$ and $B \subseteq \mathrm{SL}_n(\mathbb{R})$ Borel.

e) Let G denote the $ax+b$ group defined as

$$G = \left\{ \begin{pmatrix} a & b \\ & 1 \end{pmatrix} ; a \in \mathbb{R}^\times, b \in \mathbb{R} \right\}$$

Note that every element in G can be written in a unique fashion as a product of the form:

$$\begin{pmatrix} a & b \\ & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ & 1 \end{pmatrix}$$

where $\alpha \in \mathbb{R}^\times$ and $\beta \in \mathbb{R}$, which yields a coordinate system $\mathbb{R}^\times \times \mathbb{R} \leftrightarrow G$. Prove that

$$\mathrm{d}m(\alpha, \beta) = \frac{1}{|\alpha|} \, \mathrm{d}\alpha \, \mathrm{d}\beta$$

defines a left Haar measure on G. Calculate $\Delta_G(\alpha, \beta)$ for $\alpha \in \mathbb{R}^\times$ and $\beta \in \mathbb{R}$.

Solution. We use the coordinate system $\varphi : \mathrm{Aff}_1(\mathbb{R}) \ni (a,b) \mapsto (a, a^{-1}b) \in \mathbb{R}^\times \times \mathbb{R}$. On $\mathbb{R}^\times \times \mathbb{R}$ we define the measure $d\nu(\alpha,\beta) := \frac{1}{|\alpha|} d\alpha \, d\beta$ and we claim that $(\varphi^{-1})_* \nu$ is a left-Haar measure on $\mathrm{Aff}_1(\mathbb{R})$.

For $g \in G$ we denote as in the lecture $\lambda(g), \rho(g)$ the left, resp. right, action of g on $C_c(G)$. Let $f \in C_c\big(\mathrm{Aff}_1(\mathbb{R})\big)$ and let $g = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \in \mathrm{Aff}_1(\mathbb{R})$. Then a computation shows

$$g\varphi^{-1}(\alpha,\beta) = \begin{pmatrix} x\alpha & x\alpha\beta + y \\ 0 & 1 \end{pmatrix} = \varphi^{-1}(x\alpha, \beta + (x\alpha)^{-1}y). \tag{1}$$

and also

$$\varphi^{-1}(\alpha,\beta)g = \begin{pmatrix} x\alpha & \alpha y + \alpha\beta \\ 0 & 1 \end{pmatrix} = \varphi^{-1}(\alpha x, x^{-1}y + x^{-1}\beta). \tag{2}$$

We check left-invariance:

$$\begin{aligned}
\lambda^*(g)(\varphi_*^{-1}\nu)(f) &\overset{def}{=} (\varphi_*^{-1}\nu)(\lambda(g^{-1})f) \overset{def}{=} \nu\big((\lambda(g^{-1})f)\circ\varphi^{-1}\big) \\
&\overset{def}{=} \int_{\mathbb{R}^\times}\int_{\mathbb{R}} (\lambda(g^{-1})f)\circ\varphi^{-1}(\alpha,\beta)d\nu(\alpha,\beta) \\
&\overset{def}{=} \int_{\mathbb{R}^\times}\left(\int_{\mathbb{R}} \frac{f(g\cdot\varphi^{-1}(\alpha,\beta)}{|\alpha|}d\beta\right)d\alpha \\
&\overset{(1)}{=} \int_{\mathbb{R}^\times}\left(\int_{\mathbb{R}} \frac{f\circ\varphi^{-1}(x\alpha,\beta+(x\alpha)^{-1}y)}{|\alpha|}d\beta\right)d\alpha \\
d\beta \text{ left-invariant} \rightarrow \quad &= \int_{\mathbb{R}^\times}\left(\int_{\mathbb{R}} \frac{f\circ\varphi^{-1}(x\alpha,\beta)}{|\alpha|}d\beta\right)d\alpha \\
\text{change of variables } \psi(z,w)=(x^{-1}z,w) \rightarrow \quad &= \int_{\mathbb{R}^\times}\left(\int_{\mathbb{R}} \frac{f\circ\varphi^{-1}(z,w)}{|x^{-1}z|}|x^{-1}|dw\right)dz \\
&= \int_{\mathbb{R}^\times}\left(\int_{\mathbb{R}} \frac{f\circ\varphi^{-1}(z,w)}{|z|}dw\right)dz \\
&\overset{def}{=} \int_{\mathbb{R}^\times}\int_{\mathbb{R}} f\circ\varphi^{-1}(z,w)d\nu(z,w) \overset{def}{=} (\varphi_*^{-1}\nu)(f)
\end{aligned}$$

The modular function is determined by $\Delta_G(g)(\varphi_*^{-1}\nu)(f) = (\varphi_*^{-1}\nu)(\rho(g)f)$.
So for $f \in C_c(G), g = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ we compute

$$\begin{aligned}
(\varphi_*^{-1}\nu)(\rho(g)f) &= \nu\left(\rho(g)f\circ\varphi^{-1}\right) \\
&\overset{def}{=} \int_{\mathbb{R}^\times}\int_{\mathbb{R}} \frac{(\rho(g)f)\circ\varphi^{-1}(\alpha,\beta)}{|\alpha|}d\beta d\alpha \\
&\overset{def}{=} \int_{\mathbb{R}^\times}\int_{\mathbb{R}} \frac{f(\varphi^{-1}(\alpha,\beta)g)}{|\alpha|}d\beta d\alpha \\
&\overset{(2)}{=} \int_{\mathbb{R}^\times}\int_{\mathbb{R}} \frac{f\circ\varphi^{-1}(\alpha x, x^{-1}y+x^{-1}\beta)}{|\alpha|}d\beta d\alpha \\
d\beta \text{ left-invariant} \rightarrow \quad &= \int_{\mathbb{R}^\times}\int_{\mathbb{R}} \frac{f\circ\varphi^{-1}(\alpha x, x^{-1}\beta)}{|\alpha|}d\beta d\alpha \\
\text{change of variables } \eta(z,w)=(x^{-1}z,xw) \rightarrow \quad &= \int_{\mathbb{R}^\times}\int_{\mathbb{R}} \frac{f\circ\varphi^{-1}(z,w)}{|x^{-1}z|}dwdz \\
&= |x|\int_{\mathbb{R}^\times}\int_{\mathbb{R}} \frac{f\circ\varphi^{-1}(z,w)}{|z|}dwdz \\
&\overset{def}{=} |x|(\varphi_*^{-1}\nu)(f)
\end{aligned}$$

Therefore $\Delta_{\mathrm{Aff}_1(\mathbb{R})}(\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}) = |x|$.

Exercise 3 (Haar Measure and Transitive Actions)**.** Let G be a locally compact Hausdorff group and let X be a topological space. Suppose that G acts on X continuously and transitively. Let $o \in X$, and denote $\pi\colon G \to X, g \mapsto g \cdot o$. Further, let

$$H \coloneqq \mathrm{Stab}(o) = \{h \in G \mid h \cdot o = o\}$$

be the stabilizer of o.

Suppose there is a continuous section $\sigma\colon X \to G$ of π, i.e. $\pi \circ \sigma = \mathrm{Id}_X$.

a) Show that $\psi\colon X \times H \to G, (x, h) \mapsto \sigma(x)h$ is a homeomorphism.

   Hint: Find a continuous inverse!

   Solution. We define $\varphi\colon G \to X \times H$ via

$$\varphi(g) \coloneqq (\pi(g), \sigma(\pi(g))^{-1}g)$$

   for all $g \in G$.

   Note that

$$\sigma(\pi(g)) \cdot o = \pi(\sigma(\pi(g)) = \pi(g) = g \cdot o,$$

   whence $\sigma(\pi(g))^{-1}g \cdot o = o$ and $\sigma(\pi(g))^{-1}g \in H = \mathrm{Stab}(o)$. This shows that φ is well-defined. Moreover, φ is continuous as a composition of continuous functions.

   We will now show that φ is the inverse of ψ, i.e. $\psi \circ \varphi = \mathrm{Id}_G$ and $\varphi \circ \psi = \mathrm{Id}_{X \times H}$.

   Let $g \in G$. We compute:

$$\begin{aligned}\psi(\varphi(g)) &= \psi(\pi(g), \sigma(\pi(g))^{-1}g) \\ &= \sigma(\pi(g))\sigma(\pi(g))^{-1}g = g.\end{aligned}$$

   Let $x \in X, h \in H$. We compute:

$$\begin{aligned}\varphi(\psi(x, h)) &= \varphi(\sigma(x)h) \\ &= (\pi(\sigma(x)h), \sigma(\pi(\sigma(x)h))^{-1}\sigma(x)h) \\ &= (\sigma(x)h \cdot o, \sigma(\sigma(x)h \cdot o)^{-1}\sigma(x)h) \\ &= (\sigma(x) \cdot o, \sigma(\sigma(x) \cdot o)^{-1}\sigma(x)h) \\ &= (x, \sigma(x)^{-1}\sigma(x)h) \\ &= (x, h).\end{aligned}$$

b) Suppose there is a (left) Haar measure ν on H and suppose there is a left G-invariant Borel regular measure λ on X.

   Show that the push-forward measure $\psi_*(\lambda \otimes \nu)$ is a (left) Haar measure on G.

Solution. All we need to see is that the push-forward measure $\mu = \psi_*(\lambda \otimes \nu)$ is left G-invariant.

Let $f \in C_c(G)$ and $g_0 \in G$. We compute:

$$\begin{aligned}
\int_G f(g_0 g)\, d\mu(g) &= \int_{X\times H} f(g_0\psi(x,h))\, d(\lambda\otimes\nu)(x,h) \\
\text{(Fubini)} \quad &= \int_X \int_H f(g_0\sigma(x)h)\, d\nu(h)\, d\lambda(x) \\
&= \int_X \int_H f(\sigma(g_0\cdot x)\underbrace{\sigma(g_0\cdot x)^{-1} g_0 \sigma(x)}_{\in H} h)\, d\nu(h)\, d\lambda(x) \\
\text{(left invariance of } \nu) \quad &= \int_X \int_H f(\sigma(g_0\cdot x)h)\, d\nu(h)\, d\lambda(x) \\
\text{(left } G\text{-invariance of } \lambda) \quad &= \int_X \int_H f(\sigma(x)h)\, d\nu(h)\, d\lambda(x) \\
&= \int_G f(g)\, d\mu(g)
\end{aligned}$$

c) Find a Haar measure on $\mathrm{Iso}(\mathbb{R}^2)$.

Solution. We want to apply part b).

Note that $\mathrm{Iso}(\mathbb{R}^2)$ acts continuously and transitively on $\mathbb{R}^2$. Indeed, any translation $T_x\colon \mathbb{R}^2 \to \mathbb{R}^2, y \mapsto x+y$ ($x \in \mathbb{R}^2$) is a Euclidean isometry, that maps 0 to x.
The stabilizer of $0 \in \mathbb{R}^2$ is $\mathrm{Stab}(0) = O(2,\mathbb{R})$.

Moreover, translations give a continuous section $\sigma\colon \mathbb{R}^2 \to \mathrm{Iso}(\mathbb{R}^2), x \mapsto T_x$.

We would like to apply b) with $G = \mathrm{Iso}(\mathbb{R}^2)$, $X = \mathbb{R}^2$, $H = O(2,\mathbb{R})$. For this, we need invariant Haar measures on $\mathbb{R}$ and on $O(2,\mathbb{R})$.

- The Lebesgue measure λ on $\mathbb{R}^2$ is $\mathrm{Iso}(\mathbb{R}^2)$-invariant.
- To find an invariant Haar measure on $O(2,\mathbb{R})$ we apply part b) again.
  Observe that $O(2,\mathbb{R})$ acts transitively on the group with two elements $\{\pm 1\}$ via

  $$k * \varepsilon := \det(k)\cdot\varepsilon \text{ for every } k \in O(2,\mathbb{R}), \varepsilon \in \{\pm 1\}.$$

  We obtain a surjective map $\pi' = \det\colon O(2,\mathbb{R}) \to \{\pm 1\}, k \mapsto \det(k)\cdot 1 = \det(k)$. A section $\sigma'\colon \{\pm 1\} \to O(2,\mathbb{R})$ of det is given by

  $$\sigma'(\varepsilon) = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix},$$

  which is continuous because $\{\pm 1\}$ carries the discrete topology. The stabilizer of 1 is then $\det^{-1}(1) \cap O(2,\mathbb{R}) = \mathrm{SO}(2,\mathbb{R})$ and the usual Lebesgue measure on $[0,2\pi) \simeq \mathbb{R}/2\pi\mathbb{Z}$ pushes-forward to a left Haar measure $\xi = \varphi_*(\lambda|_{[0,2\pi)})$ on $\mathrm{SO}(2,\mathbb{R})$ via the isomorphism

  $$\begin{aligned} \varphi\colon \mathbb{R}/2\pi\mathbb{Z} &\longrightarrow \mathrm{SO}(2,\mathbb{R}), \\ \theta &\longmapsto \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \end{aligned}$$

Clearly, an invariant measure on $\{\pm 1\}$ is given by the counting measure. Applying b) to

$$\begin{aligned}
&G' = O(2,\mathbb{R}),\\
&H' = SO(2,\mathbb{R}) \text{ with } \varphi_*\lambda|_{[0,2\pi)}, \text{ and}\\
&X' = \{\pm 1\} \text{ with counting measure } \mu
\end{aligned}$$

we obtain a left Haar measure ν on $O(2,\mathbb{R})$ given by

$$\begin{aligned}
\int_{O(2,\mathbb{R})} f(k)\,d\nu(k) &= \int_{\{\pm1\}\times SO(2,\mathbb{R})} f(\psi(\varepsilon,s)d\mu\otimes\varphi_*\lambda(\varepsilon,s)\\
&= \int_{\{\pm1\}}\int_{SO(2,\mathbb{R})} f(\sigma'(\varepsilon)\cdot s)d\varphi_*\lambda(s)d\mu(\varepsilon)\\
&= \sum_{\varepsilon=\pm1}\int_0^{2\pi} f(\sigma'(\varepsilon)\cdot\varphi(\theta))d\lambda(\theta)\\
&= \sum_{\varepsilon=\pm1}\int_0^{2\pi} f\left(\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \varepsilon\sin(\theta) & \varepsilon\cos(\theta)\end{pmatrix}\right)d\theta
\end{aligned}$$

for every $f\in C_c(O(2,\mathbb{R}))$.

Putting everything together we obtain

$$\begin{aligned}
\int_{\mathrm{Iso}(\mathbb{R}^2)} f(g)\,d\mu(g) &= \int_{\mathbb{R}^2}\int_{O(2,\mathbb{R})} f(T_x k)\,d\nu(k)\,dx\\
&= \int_{\mathbb{R}^2}\sum_{\varepsilon=\pm1}\int_0^{2\pi} f\left(T_x\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \varepsilon\sin(\theta) & \varepsilon\cos(\theta)\end{pmatrix}\right)d\theta\,dx
\end{aligned}$$

Exercise 4 ($\mathrm{Aut}(\mathbb{R}^n,+)\cong \mathrm{GL}(n,\mathbb{R})$)**.** For a topological group G, we denote by $\mathrm{Aut}(G)$ the group of bijective, continuous homomorphisms of G with continuous inverse. Consider the locally compact Hausdorff group $G=(\mathbb{R}^n,+)$ where $n\in\mathbb{N}_0$.

a) Show that $\mathrm{Aut}(G)$, i.e. the group of bijective homomorphisms which are homeomorphisms as well, is given by $\mathrm{GL}_n(\mathbb{R})$.

Solution. Let $\varphi\in\mathrm{Aut}(\mathbb{R}^n)$, then φ is in particular additive and thus $\varphi(kv)=k\varphi(v)$ for all $v\in\mathbb{R}^n$, for all $k\in\mathbb{Z}$. Let $m\in\mathbb{Z}$, $n\in\mathbb{N}$ and $q=\frac{m}{n}\in\mathbb{Q}$, then

$$n\varphi(qv)=\varphi(nqv)=\varphi(mv)=m\varphi(v)\implies \varphi(q)\varphi(v)=q\varphi(v)$$

and φ is $\mathbb{Q}$-linear. $\mathbb{R}$-linearity follows from continuity of φ and thus $\varphi\in\mathrm{End}_{\mathbb{R}}(\mathbb{R}^n)$. As φ is invertible, any choice of basis realizes φ as an element in $\mathrm{GL}_n(\mathbb{R})$. It is clear that for such a choice of a basis, any $g\in\mathrm{GL}_n(\mathbb{R})$ defines an element in $\mathrm{Aut}(\mathbb{R}^n)$ and that the correspondence is 1-1 and obeys the various group structures (on $\mathrm{Aut}(G)$ and $\mathrm{GL}_n(\mathbb{R})$).

b) Show that $\mathrm{mod}:\mathrm{Aut}(G)\to\mathbb{R}_{>0}$ is given by $\alpha\mapsto|\mathrm{det}\alpha|$.

Remark. By the definition given in the lecture $\mathrm{mod}(\alpha)$ is the unique positive real number such that $m(\alpha \cdot f) = \mathrm{mod}(\alpha) m(f)$ for all $f \in C_c(G)$, m left-Haar measure on G.
This definitions may differ by an inverse from other definitions in the literature.

Solution. The n-dimensional Lebesgue measure λ_n on $\mathbb{R}^n$ clearly is a Haar measure for $\mathbb{R}^n$: it is translation invariant and

$$\lambda_n\big(B_r(v)\big) = \frac{(\sqrt{\pi} r)^n}{\Gamma(\frac{n}{2}+1)} \in (0,\infty) \quad (r > 0, v \in \mathbb{R}^n),$$

showing that it is positive on open and finite on compact subsets of $\mathbb{R}^n$. Let $f \in C_c(\mathbb{R}^n)$, $g \in \mathrm{GL}_n(\mathbb{R})$. We check that $\lambda_n(g^{-1} \cdot f) = |\det g|^{-1} \lambda_n(f)$:

$$\begin{aligned} \lambda_n(g^{-1} f) &= \int_{\mathbb{R}^n} f(gv)\, d\lambda_n(v) = \frac{1}{|\det g|} \int_{\mathbb{R}^n} f(gv)\, |\det g|\; d\lambda_n(v) \\ \text{change of variables} \rightarrow \quad &= |\det g|^{-1} \int_{\mathbb{R}^n} f(v)\, d\lambda_n(v) \\ &= |\det g|^{-1}\, \lambda_n(f). \end{aligned}$$

c) Prove that there exists a discontinuous, bijective homomorphism from the additive group $(\mathbb{R}, +)$ to itself.

Solution. Using Zorn's lemma, construct a $\mathbb{Q}$-basis of $\mathbb{R}$ containing 1. Denote this basis by $\{x_i; i \in I\}$ for any infinite index set I containing 0 such that $x_0 = 1$ (I is infinite as otherwise $\mathbb{R}$ would be algebraic over $\mathbb{Q}$). Fix $i, j \in I \setminus \{0\}$ such that $i \neq j$ and define a linear map $\varphi : \mathbb{R} \to \mathbb{R}$ by $\mathbb{Q}$-linear extension of

$$\forall k \in I : \varphi(x_k) = \begin{cases} x_j & \text{if } k = i, \\ x_i & \text{if } k = j, \\ x_k & \text{else.} \end{cases}$$

Then φ is a homomorphism by definition and is the identity on $\mathbb{Q}$. Since every real number is the limit of a $\mathbb{Q}$-Cauchy sequence[1], let $(q_n)_{n\in\mathbb{N}} \in \mathbb{Q}^{\mathbb{N}}$ Cauchy such that $\lim_{n\to\infty} q_n = x_i$, then

$$\lim_{n\to\infty} \varphi(q_n) = \lim_{n\to\infty} q_n = x_i \neq x_j = \varphi(x_i) = \varphi(\lim_{n\to\infty} q_n).$$

Exercise 5 (Iterated Quotient Measures)**.** Let G be a locally compact Hausdorff group. Show that if $H_1 \leq H_2 \leq G$ are closed subgroups and H_1, H_2, G are all unimodular then there exist invariant measures dx, dy, dz on $G/H_1, G/H_2$ and H_2/H_1 respectively such that

$$\int_{G/H_1} f(x)dx = \int_{G/H_2} \left(\int_{H_2/H_1} f(yz)dz \right) dy$$

for all $f \in C_c(G/H_1)$.

[1] For example: given $x \in \mathbb{R}$ take $q_n := \frac{\lfloor nx \rfloor}{n} \in \mathbb{Q}$, so that $\frac{nx-1}{n} \leq q_n \leq \frac{nx}{n}$.

Solution. We will use the existence of quotient measures here extensively. Note that H_1, H_2 and G are unimodular such that the necessary and sufficient condition for the existence of quotient measures is always met.

Let dg be a Haar measure on G and dh_1 a Haar measure on H_1. We have $\Delta_G|_{H_1} \equiv 1 \equiv \Delta_{H_1}$ such that there is a left-invariant measure dx on G/H_1 satisfying

$$\int_G F(g)dg = \int_{G/H_1} \int_{H_1} F(xh_1)dh_1dx,$$

for every $F \in C_c(G)$.

Let dh_2 be a Haar measure on H_2. We have $\Delta_{H_2}|_{H_1} \equiv 1 \equiv \Delta_{H_1}$ such that there is a left-invariant measure dz on H_2/H_1 satisfying

$$\int_{H_2} F(h_2)dh_2 = \int_{H_2/H_1} \int_{H_1} F(zh_1)dh_1dz,$$

for every $F \in C_c(H_2)$.

Finally, we have $\Delta_G|_{H_2} \equiv 1 \equiv \Delta_{H_2}$ such that there is a left-invariant measure dy on G/H_2 satisfying

$$\int_G F(g)dg = \int_{G/H_2} \int_{H_2} F(yh_2)dh_2dy,$$

for every $F \in C_c(G)$.

We claim that these measures satisfy the hypothesis.

Let $f \in C_c(G/H_1)$. By a lemma from the lecture we may find an $F \in C_c(G)$ such that

$$f(gH_1) = \int_{H_1} F(gh_1)dh_1.$$

We compute

$$\begin{aligned}
\int_{G/H_1} f(x)dx &= \int_{G/H_1} \int_{H_1} F(xh_1)dh_1dx \\
&= \int_G F(g)dg \\
&= \int_{G/H_2} \int_{H_2} F(yh_2)dh_2dy \\
&= \int_{G/H_2} \int_{H_2/H_1} \int_{H_1} F(yzh_1)dh_1dzdy \\
&= \int_{G/H_2} \int_{H_2/H_1} f(yz)dzdy.
\end{aligned}$$

Exercise 6 (No $\mathrm{SL}_2(\mathbb{R})$-invariant Measure on $\mathrm{SL}_2(\mathbb{R})/P$)**.** Let $G = \mathrm{SL}_2(\mathbb{R})$ and P be the subgroup of upper triangular matrices. Show directly that there is no (non-trivial) finite G-invariant measure on G/P.

Hint: Identify $G/P \cong \mathbb{S}^1 \cong \mathbb{R} \cup \{\infty\}$ with the unit circle and consider a rotation

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and a translation

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Solution. Recall that $G = \mathrm{SL}(2,\mathbb{R})$ acts on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \,|\, \Im z > 0\} \subset \hat{\mathbb{C}}$ and its boundary $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\} \subset \hat{\mathbb{C}}$ via Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.z = \frac{az+b}{cz+d}.$$

Note that $\mathrm{SL}(2,\mathbb{R})$ acts transitively on $\partial\mathbb{H}$ and the stabilizer of ∞ is the subgroup of upper triangular matrices P. We may therefore identify $G/P \cong \mathbb{R} \cup \{\infty\}$.

Suppose there is a finite G-invariant measure m on $G/P \cong \mathbb{R} \cup \{\infty\}$. Consider the restriction $\mu = m|_{\mathbb{R}}$ of this measure to the real line. Observe that G acts on the real line via translations

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.\xi = \xi + t, \qquad \xi, t \in \mathbb{R},$$

such that μ is in particular a translation invariant measure on $\mathbb{R}$, i.e. μ is a Haar measure on $\mathbb{R}$. By uniqueness of Haar measures μ must be a multiple of the Lebesgue measure on $\mathbb{R}$. Since m is finite μ is the zero measure. That means that m is a positive multiple of the dirac measure at ∞, i.e. $m = \lambda \cdot \delta_\infty$ for some $\lambda > 0$. Now consider the rotation

$$i(z) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.z = -\frac{1}{z}$$

that sends ∞ to 0. By G-invariance we must have

$$\lambda \cdot \delta_\infty = i_*(\lambda \cdot \delta_\infty) = \lambda \cdot \delta_0$$

such that $\lambda = 0$; in contradiction to our assumption.