Solutions Exercise Sheet 3

Exercise 1 (Regular Subgroups are closed)**.** Let G be a Lie group, $H \leq G$ a subgroup that is also a regular submanifold. Prove that H is a closed subgroup of G.

Solution. Let $x \in \overline{H}$. As G is clearly first countable, we find $(x_n)_{n\in\mathbb{N}} \in H^{\mathbb{N}}$ such that $x = \lim_{n\to\infty} x_n$. Let $V \subseteq W \subseteq \overline{W} \subseteq U$ open neighbourhoods of $1 \in G$ with compact closure and assume that $\psi : U \to (-1,1)^{\dim G}$ is a chart as in the definition of a regular submanifold. Assume furthermore that V is symmetric and $VV \subseteq W$. By assumption, there is $N \geq 1$ such that $x_n \in xV$ for all $n \geq N$, thus $x_N^{-1}x_n \in V^{-1}x^{-1}xV = VV \subseteq W$ for all $n \geq N$, and thus $x_N^{-1}x_n \in H \cap VV \subseteq H \cap \overline{W}$. We note that $H \cap \overline{W}$ is compact by the choice of U. Indeed, $\psi(\overline{W}) \subseteq (-1,1)^{\dim G}$ is compact, and so is $\psi(\overline{W}) \cap \{0\}^{\dim G - \dim H} \times (-1,1)^{\dim H}$. But $x_N^{-1}x_n$ is convergent and has a limit y in $H \cap \overline{W}$; whence $x_N y = x \in H$.

Exercise 2 (Non-closed Subgroup)**.** Give an example of a Lie group G and a subgroup $H < G$ that is not closed and not a Lie group with the topology induced from G.

Solution. Observe that the rational numbers $\mathbb{Q}$ are a countable dense subgroup of $\mathbb{R}$. Therefore, $\mathbb{Q}$ is not a Lie group:

Indeed, suppose $\mathbb{Q}$ is a Lie group of dimension $n \geq 0$. Then there is a chart $\varphi : (-\varepsilon,\varepsilon) \cap \mathbb{Q} \to \mathbb{R}^n$ about $0 \in \mathbb{Q}$ to some $\mathbb{R}^n$ which is a homeomorphism onto its open image $V = \varphi((-\varepsilon,\varepsilon) \cap \mathbb{Q}) \subset \mathbb{R}^n$; in particular φ is a bijection and V and $(-\varepsilon,\varepsilon) \cap \mathbb{Q}$ have the same cardinality. If $n = 0$ then $V = \{0\}$ by definition which yields a contradiction to $(-\varepsilon,\varepsilon) \cap \mathbb{Q}$ being countably infinite. On the other hand, if $n \geq 1$ then V is uncountably infinite as an open subset of $\mathbb{R}^n$ which again yields a contradiction to $(-\varepsilon,\varepsilon) \cap \mathbb{Q}$ being countably infinite.

Exercise 3 (Differential of det)**.** We consider the determinant function $\det : \mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}^*$. Show that its differential at the identity matrix I is the trace function

$$D_I \det = \mathrm{tr}.$$

Solution. Let $A \in \mathbb{R}^{n\times n} \cong T_I\,\mathrm{GL}(n,\mathbb{R})$. We compute

$$\begin{aligned}
D_I\det(A) &= \frac{d}{dt}\bigg|_{t=0} \begin{vmatrix} 1+ta_{1,1} & ta_{1,2} & \cdots & ta_{1,n} \\ ta_{2,1} & 1+ta_{2,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & ta_{n-1,n} \\ ta_{n,1} & \cdots & ta_{n,n-1} & 1+ta_{n,n} \end{vmatrix} \\
&\overset{(*)}{=} \frac{d}{dt}\bigg|_{t=0} \left((1+ta_{1,1}) \begin{vmatrix} 1+ta_{2,2} & ta_{2,3} & \cdots & ta_{2,n} \\ ta_{3,2} & 1+ta_{3,3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & ta_{n-1,n} \\ ta_{n,2} & \cdots & ta_{n,n-1} & 1+ta_{n,n} \end{vmatrix} \right) \\
&\quad + \sum_{j=2}^{n} (-1)^{j+1} \frac{d}{dt}\bigg|_{t=0} \left(ta_{2,j} \begin{vmatrix} ta_{1,2} & \cdots & ta_{1,n} \\ & * & \end{vmatrix} \right) \\
&\overset{(**)}{=} \left(a_{1,1} \begin{vmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{vmatrix} + \frac{d}{dt}\bigg|_{t=0} \begin{vmatrix} 1+ta_{2,2} & ta_{2,3} & \cdots & ta_{2,n} \\ ta_{3,2} & 1+ta_{3,3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & ta_{n-1,n} \\ ta_{n,2} & \cdots & ta_{n,n-1} & 1+ta_{n,n} \end{vmatrix} \right) \\
&\quad + \sum_{j=2}^{n} (-1)^{j+1} \left(a_{2,j} \begin{vmatrix} 0\cdot a_{1,2} & \cdots & 0\cdot a_{1,n} \\ & * & \end{vmatrix} + 0\cdot * \right) \\
&= a_{1,1} + \frac{d}{dt}\bigg|_{t=0} \det(I_{2\times n,2\times n} + tA_{2\times n,2\times n}) \\
&= \cdots = a_{1,1} + \cdots + a_{n,n} = \mathrm{tr}(A),
\end{aligned}$$

where we have developed the first column in $(*)$ and applied the product rule in $(**)$.

Exercise 4 (The Matrix Lie Groups $O(p,q)$ and $U(p,q)$)**.** Let $p, q \in \mathbb{N}$ and $n = p + q$.

a) We define the (indefinite) symmetric bilinear form $\langle \cdot, \cdot \rangle_{p,q}$ of signature (p,q) on $\mathbb{R}^n$ to be

$$\langle v, w \rangle_{p,q} := v_1 w_1 + \cdots + v_p w_p - v_{p+1} w_{p+1} - \cdots - v_{p+q} w_{p+q}$$

for all $v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_n) \in \mathbb{R}^n$. As the orthogonal group $O(n)$ is defined to be the group of matrices that preserve the standard Euclidean inner product we may now define $O(p,q)$ to be the group of matrices that preserve the above bilinear form:

$$O(p,q) := \{A \in \mathrm{GL}(n, \mathbb{R}) : \langle Av, Aw \rangle_{p,q} = \langle v, w \rangle_{p,q} \quad \forall v, w \in \mathbb{R}^n\}.$$

Show that $O(p,q)$ is a Lie group using the inverse function theorem/constant rank theorem. What is its dimension?

Solution. We define

$$I_{p,q} := \mathrm{diag}(\underbrace{1, \ldots, 1}_{p\text{-times}}, \underbrace{-1, \ldots, -1}_{q\text{-times}})$$

to be the diagonal matrix that has $+1$ in the first p entries along the diagonal and -1 in the last q entries. It is easy to see that

$$O(p,q) := \left\{A \in \mathrm{GL}(n, \mathbb{R}) : A^T I_{p,q} A = I_{p,q}\right\}.$$

Now, define

$$f : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^{n \times n}, A \mapsto A^T I_{p,q} A,$$

such that $O(p,q) = f^{-1}(I_{p,q})$. The map f is smooth as every entry of $f(A)$ is a polynomial in the entries of $A \in \mathrm{GL}(n, \mathbb{R})$.

We proceed by showing that f has constant rank. Let $X \in T_A \mathrm{GL}(n, \mathbb{R}) \cong \mathbb{R}^{n \times n}, A \in \mathrm{GL}(n, \mathbb{R})$. We compute directly

$$\begin{aligned} D_A f(X) &= \frac{d}{dt}\bigg|_{t=0} (A + tX)^T I_{p,q} (A + tX) \\ &= \frac{d}{dt}\bigg|_{t=0} \left(A^T I_{p,q} A + t \cdot X^T I_{p,q} A + t \cdot A^T I_{p,q} X + t^2 \cdot X^T I_{p,q} X\right) \\ &= X^T I_{p,q} A + A^T I_{p,q} X = (A^T I_{p,q} X)^T + A^T I_{p,q} X. \end{aligned}$$

We claim that the image consists of all symmetric matrices $\mathrm{Sym}_n(\mathbb{R}) \subset \mathbb{R}^{n \times n}$ and that $D_A f : T_A \mathrm{GL}(n, \mathbb{R}) \cong \mathbb{R}^{n \times n} \to \mathrm{Sym}_n(\mathbb{R})$ is onto. For that consider the projection

$$\begin{aligned} p : \mathbb{R}^{n \times n} &\to \mathrm{Sym}_n(\mathbb{R}), \\ X &\mapsto \tfrac{1}{2}\left(X + X^T\right). \end{aligned}$$

It is easy to check that $p \circ p = p$ and $p|_{\mathrm{Sym}_n(\mathbb{R})} = \mathrm{Id}$, such that p is onto. Since,

$$D_A f(X) = 2 \cdot p(A^T I_{p,q} X)$$

and A is invertible, $D_A f$ is also onto. Therefore, f has constant rank $\dim \mathrm{Sym}_n(\mathbb{R})$.

It follows that $O(p,q)$ is a Lie group as multiplication $m : \mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ and inversion $i : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ are smooth maps and hence restrict to smooth maps on the regular submanifold $O(p,q) \subset \mathrm{GL}(n,\mathbb{R})$.

Every symmetric matrix is uniquely determined by its entries above and on the diagonal such that

$$\dim \mathrm{Sym}_n(\mathbb{R}) = n + (n-1) + \cdots + 1 = \frac{n(n+1)}{2}.$$

The constant rank theorem then yields

$$\begin{aligned}\dim O(p,q) = \dim f^{-1}(I_{p,q}) &= \dim \mathrm{GL}(n,\mathbb{R}) - \operatorname{rank} D_A f \\ &= n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.\end{aligned}$$

b) Similarly we may define the following symmetric sesquilinear form on $\mathbb{C}^n$

$$\langle w, z\rangle_{p,q} := \bar{w}_1 z_1 + \cdots + \bar{w}_p z_p - \bar{w}_{p+1} z_{p+1} - \cdots - \bar{w}_{p+q} z_{p+q}$$

for all $w = (w_1, \dots, w_n), z = (z_1, \dots, z_n) \in \mathbb{C}^n$, and

$$U(p,q) = \{A \in \mathrm{GL}(n,\mathbb{C}) : \langle Aw, Az\rangle_{p,q} = \langle w, z\rangle_{p,q} \quad \forall w, z \in \mathbb{C}^n\}.$$

Show that $U(p,q)$ is a (real) Lie group using the inverse function theorem/constant rank theorem. What is its (real) dimension?

Solution. This is almost the same proof as for part a). It is easy to see that

$$U(p,q) := \{A \in \mathrm{GL}(n,\mathbb{C}) : A^* I_{p,q} A = I_{p,q}\}.$$

Now, define

$$f : \mathrm{GL}(n,\mathbb{C}) \to \mathbb{C}^{n\times n}, A \mapsto A^* I_{p,q} A,$$

such that $U(p,q) = f^{-1}(I_{p,q})$. The map f is smooth as every entry of $f(A)$ is a polynomial in the entries of $A \in \mathrm{GL}(n,\mathbb{C})$.

We proceed by showing that f has constant rank. Let $X \in T_A \mathrm{GL}(n,\mathbb{C}) \cong \mathbb{R}^{n\times n}, A \in \mathrm{GL}(n,\mathbb{C})$. We compute directly

$$\begin{aligned}D_A f(X) &= \frac{d}{dt}\Big|_{t=0} (A+tX)^* I_{p,q} (A+tX) \\ &= \frac{d}{dt}\Big|_{t=0} \big(A^* I_{p,q} A + t \cdot X^* I_{p,q} A + t \cdot A^* I_{p,q} X + t^2 \cdot X^* I_{p,q} X\big) \\ &= X^* I_{p,q} A + A^* I_{p,q} X = (A^* I_{p,q} X)^* + A^* I_{p,q} X.\end{aligned}$$

We claim that the image consists of all Hermitian matrices $\mathrm{Herm}_n(\mathbb{C}) \subset \mathbb{C}^{n\times n}$ and that $D_A f : T_A \mathrm{GL}(n,\mathbb{C}) \cong \mathbb{C}^{n\times n} \to \mathrm{Herm}_n(\mathbb{C})$ is onto. For that consider the projection

$$\begin{aligned}p : \mathbb{C}^{n\times n} &\to \mathrm{Herm}_n(\mathbb{C}), \\ X &\mapsto \tfrac{1}{2}(X + X^*).\end{aligned}$$

It is easy to check that $p \circ p = p$ and $p|_{\mathrm{Herm}_n(\mathbb{C})} = \mathrm{Id}$, such that p is onto. Since

$$D_A f(X) = 2 \cdot p(A^T I_{p,q} X)$$

and A is invertible, $D_A f$ is also onto. Therefore f has constant rank $\dim_{\mathbb{R}} \mathrm{Herm}_n(\mathbb{C})$.

It follows that $U(p,q)$ is a Lie group as multiplication $m : \mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(n,\mathbb{C})$ and inversion $i : \mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(n,\mathbb{C})$ are smooth maps and hence restrict to smooth maps on the regular submanifold $U(p,q) \subset \mathrm{GL}(n,\mathbb{C})$.

Every Hermitian matrix is uniquely determined by its entries above and on the diagonal. In contrast to $O(p,q)$ the entries above the diagonal can be any complex number which amounts to two real dimension for each entry. However, an entry z on the diagonal can only take on real values as $z = \bar{z}$ has to hold. Therefore, these amount to one real dimension each. All in all, we get

$$\dim \mathrm{Herm}_n(\mathbb{C}) = n + 2(n-1) + 2(n-2) + \cdots + 2 \cdot 1 = n + 2 \cdot \frac{n(n-1)}{2} = n^2.$$

The constant rank theorem then yields

$$\begin{aligned} \dim U(p,q) &= \dim f^{-1}(I_{p,q}) = \dim \mathrm{GL}(n,\mathbb{C}) - \operatorname{rank} D_A f \\ &= 2n^2 - n^2 = n^2. \end{aligned}$$

Exercise 5 (Dimension of $\mathrm{O}(n,\mathbb{R})$)**.** Show that the dimension of $\mathrm{O}(n,\mathbb{R})$ is $n(n-1)/2$.

Solution. The proof is the same as in exercise 4 for $O(p,q)$ when $p = n$ and $q = 0$.

Exercise 6 (One- and two-dimensional Lie Algebras)**.** Classify the one- and two-dimensional real Lie algebras up to Lie algebra isomorphism and realize them as Lie subalgebras of some $\mathfrak{gl}_n\mathbb{R} = \mathfrak{gl}(\mathbb{R}^n)$.

<u>Hint:</u> In dimension two one can show that if the Lie algebra is non-abelian then there is a basis X, Y such that $[X,Y] = Y$.

Solution. Let $(\mathfrak{a}, [\cdot,\cdot])$ be a real Lie algebra.

We will first deal with the one-dimensional case. Suppose $\dim \mathfrak{a} = 1$ and let X be a basis vector for $\mathfrak{a}$. Due to the anti-symmetry of the Lie bracket we have

$$[X,X] = -[X,X] = 0,$$

i.e. every one-dimensional Lie algebra is abelian. We claim that the linear map $\varphi : (\mathfrak{a}, [\cdot,\cdot]) \to (\mathbb{R}, [\cdot,\cdot])$ given by $\varphi(X) = 1$ is a Lie algebra isomorphism where the Lie bracket on $\mathbb{R}$ vanishes everywhere. Clearly, φ is an isomorphism of vector spaces and

$$[\varphi(X), \varphi(X)] = 0 = \varphi(\underbrace{[X,X]}_{=0})$$

such that φ is indeed a Lie algebra isomorphism.

In order to realize $\mathfrak{a}$ as a Lie subalgebra of some $\mathfrak{gl}_n\mathbb{R}$ we need to find a one-dimensional subalgebra of some $\mathfrak{gl}_n\mathbb{R}$ on which the commutator $[\cdot,\cdot]$ in $\mathfrak{gl}_n\mathbb{R}$ vanishes. Consider

$$\mathfrak{b} = \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} : x \in \mathbb{R} \right\} \subseteq \mathfrak{gl}_2\mathbb{R}.$$

Clearly, $\mathfrak{b}$ is a linear subspace of $\mathfrak{gl}_2\mathbb{R}$. Further, note that

$$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x \cdot y & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$$

for all $x, y \in \mathbb{R}$, such that $[X, Y] = 0$ for all $X, Y \in \mathfrak{b}$. Therefore the vector space isomorphism $\psi : \mathbb{R} \to \mathfrak{b}$ given by

$$\psi(x) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$$

is also a Lie algebra isomorphism. Thus, $\psi \circ \varphi : \mathfrak{a} \hookrightarrow \mathfrak{gl}_2(\mathbb{R})$ realizes $\mathfrak{a}$ as a Lie subalgebra of $\mathfrak{gl}_2\mathbb{R}$.

Suppose $\dim \mathfrak{a} = 2$ and let $\{X, Y\}$ be a basis of $\mathfrak{a}$. Suppose $\mathfrak{a}$ is abelian, i.e. $[X, Y] = 0$. Consider

$$\mathfrak{c} := \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in \mathbb{R} \right\} \subset \mathfrak{gl}_2\mathbb{R}$$

and the vector space isomorphism $\varphi : \mathfrak{a} \to \mathfrak{c}$ given by

$$\varphi(X) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} =: E_{11}, \quad \varphi(Y) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} =: E_{22}.$$

Note that

$$E_{11} \cdot E_{22} = 0 = E_{22} \cdot E_{11},$$

such that

$$\varphi([X, Y]) = \varphi(0) = 0 = [E_{11}, E_{22}] = [\varphi(X), \varphi(Y)].$$

Therefore, $\varphi : \mathfrak{a} \to \mathfrak{c} \subset \mathfrak{gl}_2\mathbb{R}$ is a Lie algebra isomorphism. This realizes $\mathfrak{a}$ as the subalgebra $\mathfrak{c}$ of $\mathfrak{gl}_2\mathbb{R}$ and shows that every real abelian Lie algebra is isomorphic to $\mathfrak{c}$.

Finally, suppose that $\mathfrak{a}$ is non-abelian such that

$$[X, Y] = \alpha X + \beta Y \neq 0 \tag{$\star$}$$

for some $\alpha, \beta \in \mathbb{R}$. By ($\star$) not both α and β are zero such that

$$\beta\lambda - \alpha\mu = 1$$

for some $\lambda, \mu \in \mathbb{R}$. Define

$$X' := \lambda X + \mu Y, \quad Y' := \alpha X + \beta Y = [X, Y].$$

Observe that the base change from $\{X, Y\}$ to $\{X', Y'\}$ is given by the matrix

$$\begin{pmatrix} \lambda & \alpha \\ \mu & \beta \end{pmatrix}$$

with determinant $\lambda\beta - \alpha\mu = 1$ such that $\{X', Y'\}$ is again a basis of $\mathfrak{a}$. Further,

$$\begin{aligned}[X', Y'] &= [\lambda X + \mu Y, \alpha X + \beta Y] \\ &= \lambda\beta[X, Y] + \mu\alpha[Y, X] \\ &= (\beta\lambda - \alpha\mu)[X, Y] \\ &= Y'.\end{aligned}$$

Consider the vector subspace $\mathfrak{d} \subset \mathfrak{gl}_2\mathbb{R}$ generated by the matrices

$$A := \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad C := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In fact, $\mathfrak{d}$ is a Lie subalgebra:

$$\begin{aligned}[A, C] &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = C.\end{aligned}$$

This computation also shows that the linear map $\varphi : \mathfrak{a} \to \mathfrak{d}$ given by

$$\varphi(X') = A, \quad \varphi(Y') = C$$

is a Lie algebra isomorphism (it is easily seen to be an isomorphism of vector spaces). Therefore, $\mathfrak{a}$ can be realized as the subalgebra $\mathfrak{d}$ of $\mathfrak{gl}_2\mathbb{R}$. This also proves that any real, non-abelian Lie algebra $\mathfrak{a}$ is isomorphic to $\mathfrak{d}$.

Remark: Notice that the map $\Phi : \mathfrak{gl}_2\mathbb{R} \hookrightarrow \mathfrak{gl}_n\mathbb{R}$ given by

$$\Phi(A) = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & 0 \end{array}\right)$$

is an injective Lie algebra homomorphism such that the discussed realizations of $\mathfrak{a}$ as subalgebras of $\mathfrak{gl}_2\mathbb{R}$ also amount to realizations of $\mathfrak{a}$ in any $\mathfrak{gl}_n\mathbb{R}$.

Exercise 7 (The adjoint representation ad)**.** Let V be a vector space over a field k.

a) Show that the vector space of endomorphisms

$$\mathfrak{gl}(V) \coloneqq \{A\colon V \to V \text{ linear}\}$$

is a Lie algebra with the Lie bracket given by the commutator

$$[A, B] \coloneqq AB - BA$$

for all $A, B \in \mathfrak{gl}(V)$.

Solution. One immediately verifies that $\mathfrak{gl}(V)$ is an algebra with respect to the Lie bracket. What is left to check is that the commutator satisfies the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

for all $X, Y, Z \in \mathfrak{gl}(V)$.

We leave this computation to the reader.

b) Let $\mathfrak{g}$ be a Lie algebra over k. The *adjoint representation*

$$\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

is defined as $\mathrm{ad}(X)(Y) \coloneqq [X, Y]$ for all $X, Y \in \mathfrak{g}$. Show that ad is a Lie algebra homomorphism.

Solution. It is easy to check that $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is linear. Thus, we only need to check that it preserves the Lie bracket.

We compute

$$\begin{aligned}
[\mathrm{ad}(X), \mathrm{ad}(Y)](Z) &= (\mathrm{ad}(X) \circ \mathrm{ad}(Y) - \mathrm{ad}(Y) \circ \mathrm{ad}(X))(Z) \\
&= [X, [Y, Z]] - [Y, [X, Z]] \\
&= [X, [Y, Z]] + [Y, [Z, X]] \\
\text{(Jacobi identity)} &= -[Z, [X, Y]] \\
&= [[X, Y], Z] = \mathrm{ad}([X, Y])(Z)
\end{aligned}$$

for all $X, Y, Z \in \mathfrak{g}$.

Exercise 8 (Quaternions)**.** Let $\mathbb{H} := \mathbb{R} + \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ and define – in addition to the $\mathbb{R}$ vector space structure – a multiplication on $\mathbb{H}$ by requiring:

$$\begin{aligned} \mathbf{ij} =& \mathbf{k} = -\mathbf{ji}, \\ \mathbf{jk} =& \mathbf{i} = -\mathbf{kj}, \\ \mathbf{ki} =& \mathbf{j} = -\mathbf{ik}, \\ \mathbf{i}^2 =& \mathbf{j}^2 = \mathbf{k}^2 = -1. \end{aligned}$$

The resulting skew-field is called the Hamiltonian quaternions.

a) Prove that there is a ring isomorphism:

$$\mathbb{H} \cong \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \middle| a, b \in \mathbb{C} \right\}.$$

Solution. Define

$$\Phi(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) := \begin{pmatrix} a - b\mathrm{i} & c - d\mathrm{i} \\ -c - d\mathrm{i} & a + b\mathrm{i} \end{pmatrix},$$

where i denotes the imaginary unit in $\mathbb{C}$. The map Φ is clearly $\mathbb{R}$-linear. In order to show that Φ is a homomorphism of rings, it suffices to show that Φ obeys the definition of the product on the generators $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. We leave this formal and tedious check to the reader. It is clear that the map is a bijection and hence the claim follows. For convenience, we write down the image of the generators under Φ:

$$\begin{aligned} \Phi(1) &= \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \quad \Phi(\mathbf{i}) = \begin{pmatrix} -\mathrm{i} & \\ & \mathrm{i} \end{pmatrix}, \\ \Phi(\mathbf{j}) &= \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad \Phi(\mathbf{k}) = \begin{pmatrix} & -\mathrm{i} \\ -\mathrm{i} & \end{pmatrix}. \end{aligned}$$

b) Define a Lie bracket on $\mathbb{H}$ by $[u, v] := uv - vu$. Show that $V = \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$ is a Lie ideal in $\mathbb{H}$ and that the Lie subalgebra $(V, [\cdot, \cdot])$ is isomorphic to the Lie algebra $\mathbb{R}^3$ with the cross product

$$x \times y = (x_2 y_3 - y_2 x_3, x_3 y_1 - y_3 x_1, x_1 y_2 - y_1 x_2) \quad \forall x, y \in \mathbb{R}^3$$

as a Lie bracket.

Remark: A *Lie ideal* in a Lie algebra $\mathfrak{g}$ is a Lie subalgebra $\mathfrak{i} \subseteq \mathfrak{g}$ such that $[X, Y] \in \mathfrak{i}$ for all $X \in \mathfrak{g}, Y \in \mathfrak{i}$.

Solution. By multilinearity again, it suffices to check the ideal property on generators only. That is, we show that

$$\Phi(x)\Phi(y) - \Phi(y)\Phi(x) \in \langle \Phi(\mathbf{i}), \Phi(\mathbf{j}), \Phi(\mathbf{k}) \rangle_{\mathbb{R}} =: V$$

for all $x, y \in \{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ with $y \neq 1$. Denoting by $[\,\cdot,\cdot]$ the commutator on $\mathbb{C}^{2\times 2}$, one calculates

$$\begin{aligned}
[\Phi(1), \Phi(\mathbf{i})] &= [\Phi(1), \Phi(\mathbf{j})] = [\Phi(1), \Phi(\mathbf{k})] \\
&= [\Phi(\mathbf{i}), \Phi(\mathbf{i})] = [\Phi(\mathbf{j}), \Phi(\mathbf{j})] \\
&= [\Phi(\mathbf{k}), \Phi(\mathbf{k})] = 0 \\
[\Phi(\mathbf{i}), \Phi(\mathbf{j})] &= 2\Phi(\mathbf{k}) \\
[\Phi(\mathbf{i}), \Phi(\mathbf{k})] &= -2\Phi(\mathbf{j}) \\
[\Phi(\mathbf{j}), \Phi(\mathbf{k})] &= 2\Phi(\mathbf{i})
\end{aligned}$$

This proves that V is an ideal and also shows $(V, [\,\cdot,\cdot]) \cong (\mathbb{R}^3, \times)$ by linear extension of:

$$\begin{aligned}
\Phi(\mathbf{i}) &\mapsto e_1 \\
\Phi(\mathbf{j}) &\mapsto e_2 \\
\Phi(\mathbf{k}) &\mapsto e_3
\end{aligned}$$

In order to complete the exercise, we show that any Lie algebra $(E, [\,\cdot,\cdot]_E)$ over a field $\mathbb{K}$ is isomorphic to $(E, c[\,\cdot,\cdot]_E)$ for $c \in \mathbb{K}^\times$. Define $\Psi : E \to E$ by $\Psi v := c^{-1}v$. Using bilinearity, we calculate

$$c[\Psi v, \Psi w]_E = c^{-1}[v, w]_E = \Psi[v, w]_E$$

so that $\Psi : (E, [\,\cdot,\cdot]_E) \to (E, c[\,\cdot,\cdot]_E)$ preserves the Lie bracket. It is clear that Ψ is an isomorphism of vector spaces and thus Ψ becomes an isomorphism of Lie algebras.