# Solution Exercise Sheet 4

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**Exercise 1** (Related Vector Fields)**.** Let $M$, $N$ be smooth manifolds and let $\varphi : M \to N$ be a smooth map. Recall that two vector fields $X \in \mathrm{Vect}(M)$, $X' \in \mathrm{Vect}(N)$ are called $\varphi$-related if

$$d_p\varphi(X_p) = X'_{\varphi(p)}$$

for every $p \in M$.

Show that $[X, Y]$ is $\varphi$-related to $[X', Y']$ if $X \in \mathrm{Vect}(M)$ is $\varphi$-related to $X' \in \mathrm{Vect}(N)$ and $Y \in \mathrm{Vect}(M)$ is $\varphi$-related to $Y' \in \mathrm{Vect}(N)$.

**Solution.** Let $f \in C^\infty(N)$ be a smooth function on $N$ and $X, X', Y, Y'$ vector fields on $M$ and $N$ as above.

Let $p \in M$. We compute

$$\begin{aligned}
[X', Y']_{\varphi(p)} f &= X'_{\varphi(p)}(Y'f) - Y'_{\varphi(p)}(X'f) \\
&= (d_p\varphi(X_p))(Y'f) - (d_p\varphi(Y_p))(X'f) \\
&= X_p((Y'f) \circ \varphi) - Y_p((X'f) \circ \varphi).
\end{aligned}$$

Now, note that

$$(Y'f)(\varphi(q)) = Y'_{\varphi(q)} f = (d_q\varphi(Y_q))f = Y_q(f \circ \varphi), \qquad \forall q \in M,$$

because $Y$ and $Y'$ are $\varphi$-related and analogously

$$(X'f)(\varphi(q)) = X'_{\varphi(q)} f = (d_q\varphi(X_q))f = X_q(f \circ \varphi), \qquad \forall q \in M.$$

Therefore

$$\begin{aligned}
X_p((Y'f) \circ \varphi) - Y_p((X'f) \circ \varphi) &= X_p(Y(f \circ \varphi)) - Y_p(X(f \circ \varphi)) \\
&= [X, Y]_p(f \circ \varphi) \\
&= d_p\varphi([X, Y])f.
\end{aligned}$$

This shows that

$$d_p\varphi([X, Y]) = [X', Y']_{\varphi(p)},$$

i.e. $[X, Y]$ and $[X', Y']$ are $\varphi$-related.

**Exercise 2** (Leibniz Rule)**.** Let $A, B : (-\varepsilon, \varepsilon) \to \mathbb{R}^{n\times n}$ be smooth curves and define $\varphi : (-\varepsilon, \varepsilon) \to \mathbb{R}^{n\times n}$ as the product $\varphi(t) := A(t)B(t)$. Show that

$$\varphi'(t) = A'(t)B(t) + A(t)B'(t)$$

for every $t \in (-\varepsilon, \varepsilon)$.

**Solution.** Note that the $ij$-entry of $\varphi(t)$ is

$$\varphi_{ij}(t) = \sum_{k=1}^{n} A_{ik}(t)B_{kj}(t)$$

for every $t \in (-\varepsilon, \varepsilon)$.

Differentiating each entry yields

$$\begin{aligned}\varphi'_{ij}(t) &= \sum_{k=1}^{n} A'_{ik}(t)B_{kj}(t) + \sum_{k=1}^{n} A_{ik}(t)B'_{kj}(t)\\ &= (A'(t)B(t))_{ij} + (A(t)B'(t))_{ij} \qquad \forall t \in (-\varepsilon, \varepsilon)\end{aligned}$$

such that

$$\varphi'(t) = A'(t)B(t) + A(t)B'(t)$$

as claimed.

**Exercise 3** (Some Lie Algebras)**.** (a) Let $M$, $N$ be smooth manifolds and let $f : M \to N$ be a smooth map of constant rank $r$. By the constant rank theorem we know that the level set $L = f^{-1}(q)$ is a regular submanifold of $M$ of dimension $\dim M - r$ for every $q \in N$. Show that one may canonically identify

$$T_pL \cong \ker d_pf$$

for every $p \in L = f^{-1}(q)$.

**Solution.** Since $L = f^{-1}(q)$ is a regular submanifold of $M$ we may think of the tangent space $T_pL$ as a subspace of the tangent space $T_pM$. We will first show that $T_pL \subseteq \ker d_pf$. Let $v \in T_pL$ and let $\gamma : (-\varepsilon, \varepsilon) \to L = f^{-1}(q)$ be a smooth curve in $L$ such that $\gamma(0) = p$ and $\gamma'(0) = v$. Then $f(\gamma(t)) = q$ for all $t \in (-\varepsilon, \varepsilon)$, i.e. $f \circ \gamma$ is the constant curve. It follows that

$$d_pf(v) = d_{\gamma(0)}f(\gamma'(0)) = \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) = 0.$$

In particular, $v \in \ker d_pf$ as claimed.

Finally, note that $\ker d_pf$ is a subspace of $T_pM$ of dimension

$$\dim \ker d_pf = \dim T_pM - \operatorname{rank} d_pf = \dim M - r = \dim L = \dim T_pL.$$

Therefore $T_pL$ is a linear subspace of $\ker d_pf$ of maximal dimension such that $T_pL = \ker d_pf$.

(b) Use part a) to compute the Lie algebras of the Lie groups $O(n, \mathbb{R})$, $O(p, q)$, $U(n)$, $Sp(2n, \mathbb{C})$, $B(n)$ and $N(n)$ where $B(n)$ is the group of real invertible upper triangular matrices and $N(n)$ is the subgroup of $B(n)$ with only ones on the diagonal.

**Solution.** Note that all of the listed Lie groups are subgroups of $\mathrm{GL}(n, \mathbb{K})$ that are also regular submanifolds ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). In particular the inclusion maps yield injective Lie algebra homomorphisms. This implies that the corresponding Lie algebras can be canonically identified with Lie subalgebras of $\mathfrak{gl}_n\mathbb{K}$. Hence the Lie bracket will be given by the ambient Lie bracket $[\cdot, \cdot]$ of $\mathfrak{gl}_n\mathbb{K}$. Identifying $\mathfrak{gl}_n\mathbb{K} \cong T_I \,\mathrm{GL}(n, \mathbb{K}) \cong \mathbb{K}^{n\times n}$ the Lie bracket is given by the commutator

$$[A, B] = AB - BA$$

as was proved in the lecture.

(i) $O(n, \mathbb{R})$: Consider the function $f_1 : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^{n\times n}$ given by

$$f_1(A) = A^T A$$

for every $A \in \mathrm{GL}(n, \mathbb{R})$. It is easy to check that $f_1$ has constant rank and that

$$O(n) = f_1^{-1}(I).$$

By part a)

$$\mathfrak{o}(n) := \mathrm{Lie}(O(n)) \cong T_I O(n) \cong \mathrm{ker} d_I f_1 < \mathfrak{gl}_n\mathbb{R}.$$

Let $X \in \mathbb{R}^{n\times n} \cong T_I \,\mathrm{GL}(n, \mathbb{R})$. We compute

$$\begin{aligned} d_I f_1(X) &= \frac{d}{dt}\bigg|_{t=0} (I + tX)^t (I + tX) \\ &= X^t + X \end{aligned}$$

where we have used exercise 2 in the last equality. Therefore

$$\mathfrak{o}(n) = \{X \in \mathfrak{gl}_n\mathbb{R} : X^t + X = 0\}.$$

(ii) $O(p, q)$: Consider the function $f_2 : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^{n\times n}$ given by

$$f_2(A) = A^T I_{p,q} A$$

for every $A \in \mathrm{GL}(n, \mathbb{R})$, where

$$I_{p,q} = \mathrm{diag}(\underbrace{1, \dots, 1}_{p\text{-times}}, \underbrace{-1, \dots, -1}_{q\text{-times}}).$$

It is easy to check that $f_2$ has constant rank and that

$$O(p, q) = f_2^{-1}(I_{p,q}).$$

By part a)

$$\mathfrak{o}(p, q) := \mathrm{Lie}(O(p, q)) \cong T_I O(p, q) \cong \mathrm{ker} d_I f_2 < \mathfrak{gl}_n\mathbb{R}.$$

Let $X \in \mathbb{R}^{n\times n} \cong T_I \,\mathrm{GL}(n, \mathbb{R})$. We compute

$$\begin{aligned} d_I f_2(X) &= \frac{d}{dt}\bigg|_{t=0} (I + tX)^t I_{p,q} (I + tX) \\ &= X^t I_{p,q} + I_{p,q} X \end{aligned}$$

where we have used exercise 2 in the last equality. Therefore

$$\mathfrak{o}(p, q) = \{X \in \mathfrak{gl}_n\mathbb{R} : X^t I_{p,q} + I_{p,q} X = 0\}.$$

(iii) $U(n)$: Consider the function $f_3 : \mathrm{GL}(n, \mathbb{C}) \to \mathbb{C}^{n\times n}$ given by

$$f_3(A) = A^* A$$

for every $A \in \mathrm{GL}(n, \mathbb{C})$. It is easy to check that $f_3$ has constant rank and that

$$U(n) = f_3^{-1}(I).$$

By part a)

$$\mathfrak{u}(n) := \mathrm{Lie}(U(n)) \cong T_I U(n) \cong \ker d_I f_3 < \mathfrak{gl}_n\mathbb{C}.$$

Let $X \in \mathbb{C}^{n\times n} \cong T_I\,\mathrm{GL}(n, \mathbb{C})$. We compute

$$\begin{aligned} d_I f_3(X) &= \frac{d}{dt}\bigg|_{t=0} (I + tX)^*(I + tX) \\ &= X^* + X \end{aligned}$$

where we have used exercise 2 in the last equality. Therefore

$$\mathfrak{u}(n) = \{X \in \mathfrak{gl}_n\mathbb{C} : X^* + X = 0\}.$$

(iv) $\mathrm{Sp}(2n, \mathbb{C})$: Consider the function $f_4 : \mathrm{GL}(n, \mathbb{C}) \to \mathbb{C}^{n\times n}$ given by

$$f_4(A) = A^t F A$$

for every $A \in \mathrm{GL}(n, \mathbb{C})$ where

$$F = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

It is easy to check that $f_4$ has constant rank and that

$$\mathrm{Sp}(2n, \mathbb{C}) = f_4^{-1}(F).$$

By part a)

$$\mathfrak{sp}(2n, \mathbb{C}) := \mathrm{Lie}(\mathrm{Sp}(2n, \mathbb{C})) \cong T_I\,\mathrm{Sp}(2n, \mathbb{C}) \cong \ker d_I f_4 < \mathfrak{gl}_n\mathbb{C}.$$

Let $X \in \mathbb{C}^{n\times n} \cong T_I\,\mathrm{GL}(n, \mathbb{C})$. We compute

$$\begin{aligned} d_I f_f(X) &= \frac{d}{dt}\bigg|_{t=0} (I + tX)^t F(I + tX) \\ &= X^t F + FX \end{aligned}$$

where we have used exercise 2 in the last equality. Therefore

$$\mathfrak{sp}(2n, \mathbb{C}) = \{X \in \mathfrak{gl}_n\mathbb{C} : X^t F + FX = 0\}.$$

(v) $B(n)$: Consider the function $f_5 : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^{n\times n}$ given by

$$f_5(A) = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ A_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ A_{n1} & \cdots & A_{n,n-1} & 0 \end{pmatrix}$$

for every $A \in \mathrm{GL}(n, \mathbb{R})$. It is easy to check that $f_5$ has constant rank and that

$$B(n) = f_5^{-1}(0).$$

By part a)

$$\mathfrak{b}(n) := \mathrm{Lie}(B(n)) \cong T_I B(n) \cong \ker d_I f_5 < \mathfrak{gl}_n \mathbb{R}.$$

Let $X \in \mathbb{R}^{n \times n} \cong T_I \,\mathrm{GL}(n, \mathbb{R})$. We compute

$$\begin{aligned} d_I f_5(X) &= \left.\frac{d}{dt}\right|_{t=0} f_5(I + tX) \\ &= \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ X_{21} & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ X_{n1} & \cdots & X_{n,n-1} & 0 \end{pmatrix}. \end{aligned}$$

Therefore

$$\mathfrak{b}(n) = \left\{ \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ & \ddots & \vdots \\ 0 & & X_{nn} \end{pmatrix} \in \mathbb{R}^{n \times n} \right\}.$$

(vi) $N(n)$: Consider the function $f_6 : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^{n \times n}$ given by

$$f_6(A) = \begin{pmatrix} X_{11} & & 0 \\ \vdots & \ddots & \\ X_{n1} & \cdots & X_{nn} \end{pmatrix}$$

for every $A \in \mathrm{GL}(n, \mathbb{R})$. It is easy to check that $f_6$ has constant rank and that

$$N(n) = f_6^{-1}(I).$$

By part a)

$$\mathfrak{n}(n) := \mathrm{Lie}(N(n)) \cong T_I N(n) \cong \ker d_I f_6 < \mathfrak{gl}_n \mathbb{R}.$$

Let $X \in \mathbb{R}^{n \times n} \cong T_I \,\mathrm{GL}(n, \mathbb{R})$. We compute

$$\begin{aligned} d_I f_6(X) &= \left.\frac{d}{dt}\right|_{t=0} f_6(I + tX) \\ &= \begin{pmatrix} X_{11} & & 0 \\ \vdots & \ddots & \\ X_{n1} & \cdots & X_{nn} \end{pmatrix}. \end{aligned}$$

Therefore

$$\mathfrak{n}(n) = \left\{ \begin{pmatrix} 0 & * & * \\ \vdots & \ddots & * \\ 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{n \times n} \right\}.$$

**Exercise 4** (Easy Direction of Frobenius' Theorem)**.** Let $M$ be a smooth manifold and let $\mathcal{D}$ be a distribution on $M$. Show that $\mathcal{D}$ is involutive if it is completely integrable.

**Solution.** Let $U \subset M$ be an open set and $\{X_1, \ldots, X_n\}$ a local basis of $\mathcal{D}$ defined on $U$. Further, let $q \in U$ and suppose $q$ is contained in an integral submanifold $\varphi : N \hookrightarrow M$ of $\mathcal{D}$ such that $d_p\varphi(T_pN) = \mathcal{D}_p$ for every $p \in N$, where $\varphi : N \hookrightarrow M$ is an injective immersion. Let $p \in \varphi^{-1}(q)$ and choose open neighborhoods $V' \subset N$ about $p$ and $U' \subset U$ about q such that $\varphi|_{V'} : V' \to U'$ is a smooth embedding. By using a local slice chart it is easy to see that the vector fields $\{Y_1, \ldots, Y_n\}$ defined via

$$d_{p'}\varphi(Y_i) = (X_i)_{\varphi(p')} \qquad \forall p' \in V' \, \forall i = 1, \ldots, n \tag{$\star\star$}$$

are smooth vector fields on $V' \subset N$. Here we have used that $\{(X_1)_{\varphi(p')}, \ldots, (X_n)_{\varphi(p')}\}$ is a basis of $\mathcal{D}_{\varphi(p')} = d_{p'}\varphi(T_{p'}N)$ and that the differential of $d_{p'}\varphi$ is injective for every $p' \in V'$. Note that ($\star\star$) means that $Y_i$ is $\varphi$-related to $X_i$ for every $i = 1, \ldots, n$. By exercise 1 also $[Y_i, Y_j]$ is $\varphi$-related to $[X_i, X_j]$, i.e.

$$[X_i, X_j]_{\varphi(p')} = d_{p'}\varphi[Y_i, Y_j]_{p'},$$

for every $i, j = 1, \ldots, n$. Because $\{Y_1, \ldots, Y_n\}$ are smooth vector fields on $V' \subset N$ also $[Y_i, Y_j]_{p'}$ is a smooth vector field on $V' \subset N$. This implies that $[X_i, X_j]_{\varphi(p')} \in d_{p'}\varphi(T_{p'}N) = \mathcal{D}_{\varphi(p')}$ for every $p' \in V'$; in particular $[X_i, X_j]_q \in \mathcal{D}_q$. Therefore $\mathcal{D}$ is involutive.

**Exercise 5** (Distributions and Lie Subalgebras)**.** a) Let $M$ be a smooth manifold, $X, Y \in \mathrm{Vect}(M)$ vector fields on $M$, and $f, g \in C^\infty(M)$ smooth functions. Show that

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X.$$

**Solution.** Let $h \in C^\infty(M)$ and $p \in M$. We compute

$$\begin{aligned}
([fX, gY]_p h) &= f(p)X_p(g(Yh)) - g(p)Y_p(f(Xh)) \\
&= f(p)(X_p g)(Y_p h) + f(p)g(p)X_p(Yh) \\
&\quad - g(p)(Y_p f)(X_p h) - g(p)f(p)Y_p(Xh) \\
&= f(p)g(p)([X, Y]_p h) + f(p)(X_p g)(Y_p h) - g(p)(Y_p f)(X_p h).
\end{aligned}$$

b) Show that the Lie algebra $\mathfrak{h}$ of a Lie subgroup $H$ of a Lie group $G$ determines a left-invariant involutive distribution.

<u>Remark:</u> Part a) is not necessarily needed for part b).

**Solution.** Let $\iota : H \hookrightarrow G$ be a Lie subgroup and let $X_1, \ldots, X_n$ be a basis of $T_eH \cong \mathfrak{h}$. We define smooth left-invariant vector fields $Y_1, \ldots, Y_n$ on $G$ via

$$(Y_i)_g = d_e L_g(d_e\iota \, X_i)$$

for every $g \in G$, $i = 1, \ldots, n$. These clearly define a global basis of the left-invariant distribution $\mathcal{D} = \mathrm{span}\{Y_1, \ldots, Y_n\} \subset TG$ on $G$.

We need to see that $\mathcal{D}$ is involutive. Observe that $Y_i$ is $L_g$-related to itself for every $g \in G$ by definition. By exercise 1 also $[Y_i, Y_j]$ is $L_g$-related to itself such that

$$[Y_i, Y_j]_g = [Y_i, Y_j]_{L_g(e)} = d_e L_g([Y_i, Y_j]_e)$$

for every $g \in G$. Further $Y_i$ is $\iota$-related to $X_i$ by definition. Therefore also $[Y_i, Y_j]$ is $\iota$-related to $[X_i, X_j]$ such that

$$[Y_i, Y_j]_e = [Y_i, Y_j]_{\iota(e)} = d_e\iota[X_i, X_j]_e \in \mathcal{D}_e.$$

Hence,

$$[Y_i, Y_j]_g = d_eL_g([Y_i, Y_j]_e) \in d_eL_g(\mathcal{D}_e) = \mathcal{D}_g$$

by left-invariance. This shows that $\mathcal{D}$ is involutive.

**Exercise 6** (Functions with values in immersed submanifolds)**.** Let $M', M, N$ be smooth manifolds and let $\iota\colon N \hookrightarrow M$ be an injective immersion, i.e. $\iota$ is an injective smooth map whose differential is injective. Further, let $f\colon M' \to M$ be a smooth map with $f(M) \subseteq \iota(N)$.

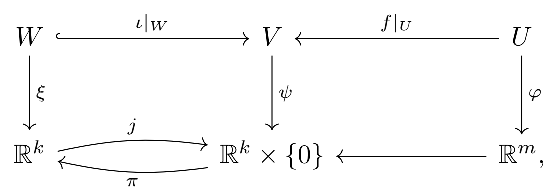
Show that $\iota^{-1} \circ f\colon M' \to N$ is smooth if it is continuous.

**Solution.** Let $x \in M'$, let $y = f(x) \in M$ and let $z = \iota^{-1}(y) \in N$. Because $\iota$ is an immersion there are open neighborhoods $W \subseteq N, V \subseteq M$ about $z, y$ resp. and charts $\xi\colon W \to \mathbb{R}^k$, $\psi\colon V \to \mathbb{R}^n$ such that $\iota(W) \subseteq V$ and

$$j(x_1, \dots, x_k) := (\psi \circ \iota \circ \xi^{-1})(x_1, \dots, x_k) = (x_1, \dots, x_k, 0, \dots, 0) \in \mathbb{R}^k \times \{0\} \subseteq \mathbb{R}^n$$

for all $(x_1, \dots, x_k) \in \mathbb{R}^k$, i.e. there is a slice chart for $N$.

Moreover, consider the open set $(\iota^{-1} \circ f)^{-1}(W) = f^{-1}(\iota(W)) \subseteq M'$ which contains an open neighborhood $U$ of $x \in M'$ with a chart $\varphi\colon U \to \mathbb{R}^m$. Because $f(U) \subseteq \iota(W) \subseteq V$, we have




where $\pi\colon \mathbb{R}^n \to \mathbb{R}^k$ is the projection on the first $k$-coordinates. This shows that $\iota^{-1} \circ f|_U$ is smooth in local charts about $x$ and $z = \iota^{-1}(f(x))$. Because $x \in M'$ was arbitrary, this shows that $\iota^{-1} \circ f$ is smooth.

**Exercise 7** (Covering maps of Lie Groups)**.** Let $G$ be a Lie group, let $H$ be a simply connected topological space and let $p : H \to G$ be a covering map.

a) Show that there is a unique Lie group structure on $H$ such that $p$ is a smooth covering and a group homomorphism. Show also that the kernel of $p$ is a discrete subgroup of $H$.

   *Recall:* $p : H \to G$ is a smooth covering if it is a topological covering which is smooth and such that each point in $G$ has a neighbourhood $U$ such that each component of $p^{-1}(U)$ is mapped diffeomorphically onto $U$ by $p$.

**Solution.** Existence:

We equip $H$ with a smooth structure such that $p$ becomes a smooth covering map. Let $U \subset H$ be an open set such that $p|_U : U \to p(U)$ is a homeomorphism and let $(V, \psi)$ be a chart of $G$ with $V \subset p(U)$. Up to shrinking $U$ assume that $U = p^{-1}(V)$. Then $(U, \psi \circ p|_U))$ is a chart of $H$. If two such charts $\psi_1 \circ p|_{U_1}$, $\psi_2 \circ p|_{U_2}$overlap, then the transition map is $\psi_1 \circ \psi_2^{-1}$, which is smooth.

Let $U \subset H$ be such that $p|_U : U \to p(U)$ is a homeomorphism and let $(V, \psi \circ p|_V)$ be a chart in $U$ and $(W, \tilde{\psi})$ be a chart in $p(U)$ such that $p(V) \subset W$. Then

$$\tilde{\psi} \circ p|_V \circ (\psi \circ p|_V)^{-1} = \tilde{\psi} \circ \psi^{-1}$$

is smooth. This shows that $p|_U$ is smooth and hence $p$ is smooth. Analogously, let $(W, \tilde{\psi})$ be a chart in $p(U)$ and $(V, \psi \circ p|_V)$ be a chart in $U$ such that $p|_U^{-1}(W) \subset V$. Then

$$(\psi \circ p|_V) \circ p|_V^{-1} \circ \tilde{\psi}^{-1} = \psi \circ \tilde{\psi}^{-1}$$

is smooth. This shows that $p|_U^{-1}$ is smooth, hence $p$ is smooth covering map.

It is not hard to verify that with $p : H \to G$ also $p \times p : H \times H \to G \times G$ is a smooth covering map. In particular, since $H$ is simply connected also $H \times H$ is simply connected such that $p \times p : H \times H \to G \times G$ is a universal covering. We will now lift the multiplication and inversion maps to $H$ and show that they define a group structure on $H$.

Let $m : G \times G \to G$ and $i : G \to G$ denote the multiplication and inversion maps of $G$, respectively, and let $\tilde{e}$ be an arbitrary element of the fiber $p^{-1}(e) \subseteq H$. Since $p \times p : H \times H \to G \times G$ is a universal covering the map $m \circ (p \times p) : H \times H \to G$ has a unique lift $\tilde{m} : H \times H \to H$ satisfying $\tilde{m}(\tilde{e}, \tilde{e}) = \tilde{e}$ and $p \circ \tilde{m} = m \circ (p \times p)$:

$$\begin{array}{ccc} H \times H & \xrightarrow{\tilde{m}} & H \\ \downarrow{\scriptstyle p \times p} & & \downarrow{\scriptstyle p} \\ G \times G & \xrightarrow{m} & G \end{array}$$

Because $p$ is a local diffeomorphism and $p \circ \tilde{m} = m \circ (p \times p)$ is smooth also $\tilde{m}$ is smooth. By the same reasoning, $i \circ p : H \to G$ has a smooth lift $\tilde{i} : H \to G$ satisfying $\tilde{i}(\tilde{e}) = \tilde{e}$ and $p \circ \tilde{i} = i \circ p$:

$$\begin{array}{ccc} H & \xrightarrow{\tilde{i}} & H \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ G & \xrightarrow{i} & G \end{array}$$

We define multiplication and inversion in $H$ by $xy = \tilde{m}(x, y)$ and $x^{-1} = \tilde{i}(x)$. By the above commutative diagrams we obtain

$$p(xy) = p(x)p(y), \qquad p(x^{-1}) = p(x)^{-1}.$$

It remains to show that $H$ is a group with these operations, for then it is a Lie group because $\tilde{m}$ and $\tilde{i}$ are smooth and the above relations imply that $p$ is a homomorphism.

First we show that $\tilde{e}$ is an identity for multiplication in $H$. Consider the map $f : H \to H$ definbed by $f(x) = \tilde{e}x$. Then

$$p(f(x)) = p(\tilde{e})p(x) = ep(x) = p(x),$$

so $f$ is a lift of $p : H \to G$. The identity map $\mathrm{Id}_H$ is another lift of $p$, and it agrees with $f$ at a point because $f(\tilde{e}) = \tilde{m}(\tilde{e}, \tilde{e}) = \tilde{e}$, so the unique lifting property of covering maps implies that $f = \mathrm{Id}_H$, or equivalently, $\tilde{e}x = x$ for all $x \in H$. The same argument shows that $x\tilde{e} = x$.

Next, to show that multiplication in $H$ is associative, consider the two maps $\alpha_L, \alpha_R : H \times H \times H \to H$ defined by

$$\alpha_L(x, y, z) = (xy)z, \quad \alpha_R(x, y, z) = x(yz).$$

Then

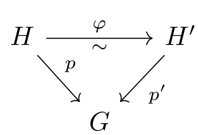
$$p(\alpha_L(x, y, z)) = (p(x)p(y))p(z) = p(x)(p(y)p(z)) = p(\alpha_R(x, y, z)),$$

so $\alpha_L$ and $\alpha_R$ are both lifts of the same map $\alpha(x, y, z) = p(x)p(y)p(z)$. Because $\alpha_L$ and $\alpha_R$ agree at $(\tilde{e}, \tilde{e}, \tilde{e})$, they are equal. A similar argument shows that $x^{-1}x = xx^{-1} = \tilde{e}$, so $\tilde{G}$ is a group.

Finally, we need to see that $\ker p$ is a discrete subgroup. To this end choose an open neighborhood $U \subseteq G$ of $e \in G$ such that $p^{-1}(U)$ is the disjoint union of open subsets $\{V_i\}_{i \in I}$ and $p|_{V_i} : V_i \to U$ is a diffeomorphism. In particular, $\ker p = p^{-1}(e) \subseteq \bigsqcup_{i \in I} V_i$ and every $x \in \ker p$ is contained in only one of the $V_i$. Hence, $\tilde{e} \in V_{i_0}$ for some $i_0 \in I$ and $(\ker p \setminus \{\tilde{e}\}) \cap V_{i_0} = \emptyset$ such that $\tilde{e}$ is an isolated point in $\ker p$. This implies that $\ker p$ is a discrete subgroup of $H$.

Uniqueness:

We first show uniqueness of the smooth structure. To this end suppose that $H$ is equipped with a Lie group structure such that $p : H \to G$ is a smooth covering and a group homomorphism. Let $(V, \varphi)$ be a chart in a smooth atlas of $H$ and, up to shrinking $V$, suppose that $p|_V : V \to p(V)$ is a diffeomorphism. Then $(p(V), \varphi \circ p|_V^{-1})$ is a smooth chart of $G$. Hence $\varphi$ comes form a chart $(U, \psi)$ of $G$, in the sense that $\psi = \varphi \circ p$ (with $\varphi = \varphi \circ p|_V^{-1}$).
We now show that the group structure is unique. Let $H'$ be the topological space $H$ equipped with another Lie group structure such that $p = p' : H' \to G$ is a smooth covering homomorphism. Because both $H$ and $H'$ are simply connected they are both universal coverings of $G$. Therefore there is a diffeomorphism $\varphi : H \to H'$ sending the neutral element $e$ of $H$ to the neutral element $e'$ of $H'$ such that the following diagram commutes.




We will now show that $\varphi : H \to H'$ is indeed a homomorphism such that $H$ and $H'$ are isomorphic Lie groups. To this end consider the set

$$A = \{(h, g) \in H \times H : \varphi(hg^{-1}) = \varphi(h)\varphi(g)^{-1}\}.$$

Clearly, $(e,e) \in A$ whence $A \neq \emptyset$. Further $A$ is closed since multiplication, inversion and $\varphi$ are all continuous maps. If we can prove that $A$ is open then $A = H \times H$ because $H \times H$ is connected, i.e. $\varphi$ is a homomorphism.

Let $(h_0, g_0) \in A$. Further, let $U' \subseteq H'$ be an open neighborhood about $\varphi(h_0 g_0^{-1}) = \varphi(h_0)\varphi(g_0)^{-1}$ such that $p'|_{U'}$ is a diffeomorphism. Let $U \subseteq H \times H$ be an open neighborhood about $(h_0, g_0)$ such that $\varphi(hg^{-1}) \in U'$ and $\varphi(h)\varphi(g)^{-1} \in U'$ for all $(h,g) \in U$; this is possible because all maps are again continuous. Then

$$\begin{aligned} p'(\varphi(hg^{-1})) &= p(hg^{-1}) = p(h)p(g)^{-1} = p'(\varphi(h))p'(\varphi(g))^{-1} \\ &= p'(\varphi(h)\varphi(g)^{-1}) \end{aligned}$$

for all $h, g \in U$ where we have used that $p$ and $p'$ are homomorphisms. By construction $\varphi(hg^{-1}), \varphi(h)\varphi(g)^{-1} \in U'$ and since $p'|_{U'}$ is bijective we get $\varphi(hg^{-1}) = \varphi(h)\varphi(g)^{-1}$ for all $h, g \in U$. Hence, $(h_0, g_0) \in U \subseteq A$ and $A$ is open because $(h_0, g_0) \in A$ were arbitrary.

It follows that $\varphi : H \to H'$ is a Lie group isomorphism.

b) Show that $p$ is a local isomorphism of Lie groups and that $dp$ is an isomorphism of Lie algebras when $H$ is equipped with the Lie group structure from part a).

**Solution.** Note that $dp$ is a Lie algebra homomorphism since $p$ is a smooth homomorphism. Because $p$ is additionally a smooth covering map there are open neighborhoods $U \subseteq G$ of $e$ and $V \subseteq H$ of $\tilde{e}$ such that $p|_V : V \to U$ is a diffeomorphism. In particular, $dp : T_{\tilde{e}}H \cong \mathfrak{h} \to T_eG \cong \mathfrak{g}$ is bijective such that $dp$ is a Lie algebra isomorphism.

c) Let $H$, $G$ be arbitrary Lie groups and let $G$ be connected. Further, let $\varphi : H \to G$ be a Lie group homomorphism. Show that $\varphi$ is a covering map if and only if $d\varphi : \mathfrak{h} \to \mathfrak{g}$ is an isomorphism.

**Solution.** First suppose that $\varphi$ is a covering map. The same proof as for part b) applies here such that $d\varphi$ is indeed an isomorphism.

Now, assume that $\varphi : H \to G$ is a smooth homomorphism such that $d\varphi : \mathfrak{h} \to \mathfrak{g}$ is an isomorphism. This means that $d_{\tilde{e}}\varphi : T_{\tilde{e}}H \to T_eG$ is invertible such that by the inverse function theorem there are open neighborhoodes $U \subseteq G$ about $e \in G$ and $V \subseteq H$ about $\tilde{e} \in H$ such that $\varphi|_V : V \to U$ is a diffeomorphism. Because $G$ is connected the open neighborhood $U$ about $e \in G$ generates $G$ and it follows easily that $\varphi : H \to G$ is surjective.

Now, choose a symmetric open neighborhood $W \subseteq V$ about $\tilde{e} \in H$ such that $W^2 \subseteq V$. Consider the open subset $U' := \varphi(W) \subseteq U$. We claim that $\varphi^{-1}(U') = \bigsqcup_{h \in \ker\varphi} Wh$ and $\varphi|_{Wh} : Wh \to U'$ is a diffeomorphism for all $h \in \ker\varphi$. Because $h \in \ker\varphi$ we have that $\varphi \circ R_h = \varphi$. Further $\varphi : W \to U'$ is a diffeomorphism such that also $\varphi : Wh \to U'$ is a diffeomorphism. Also,

$$\begin{aligned} x \in \varphi^{-1}(U') = \varphi^{-1}(\varphi(W)) &\iff \varphi(x) \in \varphi(W) \\ \iff \exists w \in W : \varphi(x) = \varphi(w) &\iff \exists w \in W : \varphi(w^{-1}x) = e \\ \iff \exists w \in W : w^{-1}x \in \ker\varphi &\iff x \in \bigcup_{h \in \ker\varphi} Wh, \end{aligned}$$

such that $\varphi^{-1}(U') = \bigcup_{h \in \ker\varphi} Wh$. Finally, if $Wh \cap Wh' \neq \emptyset$ for some $h, h' \in \ker\varphi$ then there are $w, w' \in W$ such that $wh = w'h'$, i.e. $h^{-1}h' \in W^2 \subseteq V$. Because $\varphi|_V : V \to U$ is injective and also $\varphi(h^{-1}h') = \varphi(h^{-1})\varphi(h') = e$ it follows that $h^{-1}h' = \tilde{e}$, or equivalently $h = h'$. Thus, $\bigcup_{h \in \ker\varphi} Wh$ is a disjoint union as claimed.

Using this together with the fact that $\varphi$ is a homomorphism proves that $\varphi$ is a covering map.