Solution Exercise Sheet 5

Exercise 1 (Discrete Subgroups of $\mathbb{R}^n$)**.** Let $D < \mathbb{R}^n$ be a discrete subgroup. Show that there are $x_1, \ldots, x_k \in D$ such that

- $x_1, \ldots, x_k$ are linearly independent over $\mathbb{R}$, and
- $D = \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_k$, i.e. $x_1, \ldots, x_k$ generate D as a $\mathbb{Z}$-submodule of $\mathbb{R}^n$.

Solution. We will prove this by induction on the dimension n.

Let $n = 1$ and let $D < \mathbb{R}$ be a discrete subgroup. Without loss of generality we may assume that $D \neq \{0\}$. Since D is discrete there is $x_1 \in D \setminus \{0\}$ such that $|x_1| = \min\{|x| : x \in D \setminus \{0\}\}$. We claim that $D = \mathbb{Z}x_1$. Suppose there is $y \in D \setminus \mathbb{Z}x_1$. Then there is $k \in \mathbb{Z}$ such that

$$k \cdot x_1 < y < (k+1) \cdot x_1.$$

It follows that $y - k \cdot x_1 \in D$ and $|y - k \cdot x_1| < |x_1|$ which contradicts the minimality of x_1. This shows that $D = \mathbb{Z}x_1$ and finishes the proof of the base case $n = 1$.

Let $n \in \mathbb{N}$ and assume the statement holds for all discrete subgroups of $\mathbb{R}^{n-1}$. Let $D < \mathbb{R}^n$ be a discrete subgroup. Without loss of generality we may assume that $D \neq \{0\}$. There is $x_1 \in D \setminus \{0\}$ such that $\|x_1\| = \min\{\|x\| : x \in D \setminus \{0\}\}$. Consider the quotient $\mathbb{R}^n/\mathbb{R}x_1 \cong \mathbb{R}^{n-1}$ and the projection

$$\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^n/\mathbb{R} \cdot x_1 \cong \mathbb{R}^{n-1}$$

onto it.

We claim that $D' = \pi(D) < \mathbb{R}^{n-1}$ is a discrete subgroup. We will see this by showing that $V' := \pi(B_r(0))$ is an open neighborhood of $0 \in D'$ such that $V' \cap D' = \{0\}$ where $r := \inf\{\|t \cdot x_1 - y\| : t \in \mathbb{R}, y \in D \setminus \mathbb{Z}x_1\}$.

First of all, we need to see that r is in fact positive. In order to prove this let us verify that

$$r = \inf\{\|t \cdot x_1 - y\| : t \in \mathbb{R}, y \in D \setminus \mathbb{Z}x_1\} = \inf\{\|t \cdot x_1 - y\| : t \in [0,1], y \in D \setminus \mathbb{Z}x_1\}.$$

Clearly, the left-hand-side is less than or equal to the right-hand-side. On the other hand, if $R \geq 0$ such that there are $t \in \mathbb{R}$ and $y \in D \setminus \mathbb{Z}x_1$ satisfying $R \geq \|t \cdot x_1 - y\|$ then also

$$R \geq \|t \cdot x_1 - y\| = \|(t - \lfloor t \rfloor)x_1 - (y - \lfloor t \rfloor x_1)\|;$$

whence there are $s := t - \lfloor t \rfloor \in [0,1]$ and $w := (y - \lfloor t \rfloor x_1) \in D \setminus \mathbb{Z}x_1$ such that $R \geq \|s \cdot x_1 - w\|$. Therefore, the right-hand-side is also less than or equal to the left-hand-side such that they must be equal. Because $\{t\cdot x_1 : t\in[0,1]\}\subset\mathbb{R}^n$ is compact and $D\setminus\mathbb{Z}x_1$ is discrete the infimum on the right-hand-side is in fact a minimum. It is attained at some $t_0\cdot x_1$ and $y_0\in D\setminus\mathbb{Z}x_1$. If $r=\|t_0\cdot x_1-y_0\|=0$ then $y_0=t_0x_1$ and $t_0\in(0,1)$ because $y_0\notin\mathbb{Z}x_1$. But then $\|y_0\|=t_0\|x_1\|<\|x_1\|$ which contradicts the minimality of $\|x_1\|$; whence $r>0$.

Clearly, $\pi:\mathbb{R}^n\to\mathbb{R}^{n-1}$ is an open map such that $V'=\pi(B_r(0))$ is an open neighborhood of $0\in\mathbb{R}^{n-1}$. Now, let $x'\in D'\cap V'$, i.e. $x'=\pi(u)=\pi(y)$ for some $u\in B_r(0)$, $y\in D$. Then $y-u\in\mathbb{R}x_1$, i.e. $y-u=t\cdot x_1$ for some $t\in\mathbb{R}$. This implies that

$$\|y-t\cdot x_1\|=\|u\|<r=\inf\{\|y-t\cdot x_1\| : t\in\mathbb{R}, y\in D\setminus\mathbb{Z}x_1\}.$$

We deduce that $y\in\mathbb{Z}x_1\subset\mathbb{R}x_1$; whence $x'=\pi(y)=0$ and $V'\cap D'=\{0\}$. Therefore, 0 is an isolated point in D' such that D' is a discrete subgroup of $\mathbb{R}^{n-1}$ as claimed.

By the induction hypothesis there are $x'_2,\ldots,x'_k\in D'<\mathbb{R}^{n-1}$ which are linearly independent over $\mathbb{R}$ and generate D' as a $\mathbb{Z}$-submodule, i.e. $D'=\mathbb{Z}x'_2\oplus\cdots\oplus\mathbb{Z}x'_k$. We choose for every x'_i a preimage $x_i\in\pi^{-1}(x'_i)\cap D$. These $x_1,x_2,\ldots,x_k\in D$ are linearly independent over $\mathbb{R}$ and satisfy $D=\mathbb{Z}x_1\oplus\cdots\oplus\mathbb{Z}x_k$. Indeed, let $\lambda_1,\ldots,\lambda_k\in\mathbb{R}$ such that

$$\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_kx_k=0. \tag{1}$$

Then

$$\begin{aligned}0&=\pi(\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_kx_k)\\&=\underbrace{\lambda_1\pi(x_1)}_{=0}+\lambda_2\pi(x_2)+\cdots+\lambda_k\pi(x_k)\\&=\lambda_2x'_2+\cdots+\lambda_kx'_k.\end{aligned}$$

Because $x'_2,\ldots,x'_k$ are linearly independent, $\lambda'_2=\ldots=\lambda'_k=0$. By (1), $\lambda_1x_1=0$. Finally, since $x_1\neq0$ also $\lambda_1=0$.

In order to see that $x_1,\ldots,x_k$ generate D as a $\mathbb{Z}$-module, let $y\in D$. Then

$$\pi(y)=a_2x'_2+\cdots+a_kx'_k=a_2\pi(x_2)+\cdots+a_k\pi(x_k)$$

for some $a_2,\ldots,a_k\in\mathbb{Z}$ since $x'_2,\ldots,x'_k$ generate D' as a $\mathbb{Z}$-module. Considering $y'=a_2x_2+\cdots+a_kx_k\in D$ we obtain

$$\pi(y')=\pi(a_2x_2+\cdots+a_kx_k)=a_2\pi(x_2)+\cdots+a_k\pi(x_k)=\pi(y)$$

by linearity such that $y-y'\in D\cap\ker\pi=D\cap\mathbb{R}x_1$.

We claim that $D\cap\ker\pi=\mathbb{Z}x_1$. It is immediate that $\mathbb{Z}x_1\subseteq D\cap\ker\pi$. To see the other inclusion suppose that there is $t\cdot x_1\in D$ for some $t\in\mathbb{R}\setminus\mathbb{Z}$. Then $w=(t-\lfloor t\rfloor)\cdot x_1\in D\setminus\{0\}$ and

$$\|w\|=(t-\lfloor t\rfloor)\cdot\|x_1\|<\|x_1\|$$

in contradiction to the minimality of x_1.

Therefore, $y - y' \in \mathbb{Z}x_1$ and there exists $a_1 \in \mathbb{Z}$ such that

$$y = a_1 x_1 + y' = a_1 x_1 + a_2 x_2 + \cdots + a_k x_k.$$

Hence, $D = \mathbb{Z}x_1 \oplus \cdots \oplus \mathbb{Z}x_k$.

Exercise 2 (Surjectivity of the matrix exponential). a) Show that the exponential map of $\mathrm{GL}(2,\mathbb{C})$ is surjective.

Solution. Let $A \in \mathrm{GL}(2,\mathbb{C})$. By the Jordan Normal Form theorem there is an invertible matrix P such that $PAP^{-1} =: J$ is in Jordan Normal Form. That is, J is either diagonal or

$$J = \begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix}$$

for some $z \in \mathbb{C}$. We have seen that $\exp_{\mathrm{GL}(2,\mathbb{C})} = \mathrm{Exp}$ is the matrix exponential.
It follows directly from the definition that for any $X \in \mathfrak{gl}(2,\mathbb{C})$: $\mathrm{Exp}(PXP^{-1}) = P\,\mathrm{Exp}(X)P^{-1}$.
Thus it suffices to show that any J as above is in the image of the exponential map.

If J is diagonal, $J = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}$, then there are $w_1, w_2 \in \mathbb{C}$ such that $e^{w_1} = z_1$, $e^{w_2} = z_2$ and

$$\mathrm{Exp}\begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix} = \sum_{j=0}^{\infty} \frac{1}{j!} \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}^j = \sum_{j=0}^{\infty} \begin{pmatrix} \frac{1}{j!} w_1^j & 0 \\ 0 & \frac{1}{j!} w_2^j \end{pmatrix} = \begin{pmatrix} e^{w_1} & 0 \\ 0 & e^{w_2} \end{pmatrix} = J$$

To find a preimage of $J = \begin{pmatrix} z & 1 \\ 0 & z \end{pmatrix}$ we compute $\mathrm{Exp}\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$. By induction one shows that $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^j = \begin{pmatrix} a^j & ba^{j-1} \\ 0 & a^j \end{pmatrix}$ and therefore an analogous computation as above shows that

$$\mathrm{Exp}\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} e^a & be^a \\ 0 & e^a \end{pmatrix}$$

Let $w \in \mathbb{C}$ be such that $e^w = z$, then $\mathrm{Exp}\begin{pmatrix} w & e^{-w} \\ 0 & w \end{pmatrix} = J$.

b) Show that the exponential map of $\mathrm{U}(n)$ is surjective.

Solution. Let $K = \mathrm{U}(n)$ and $\mathfrak{k} = \mathrm{Lie}\,\mathrm{U}(n) = \{X \in \mathfrak{gl}(n,\mathbb{C}) : X + X^* = 0\}$. Consider $g \in K$. Then it can be unitarily diagonalized, that is, there is $h \in K$ with $hgh^* = \begin{pmatrix} \xi_1 & & 0 \\ & \ddots & \\ 0 & & \xi_n \end{pmatrix}$ and $|\xi_i| = 1$, $i = 1, \dots, n$. Write $\xi_i = e^{ix_i}$, $x_i \in \mathbb{R}$, so that $hgh^* = \mathrm{Exp}\begin{pmatrix} ix_1 & & 0 \\ & \ddots & \\ 0 & & ix_n \end{pmatrix}$.

Thus

$$g = \mathrm{Exp}(h^* \begin{pmatrix} ix_1 & & 0 \\ & \ddots & \\ 0 & & ix_n \end{pmatrix} h).$$

And also,

$$(h^* \begin{pmatrix} ix_1 & & 0 \\ & \ddots & \\ 0 & & ix_n \end{pmatrix} h)^* = h^* \begin{pmatrix} -ix_1 & & 0 \\ & \ddots & \\ 0 & & -ix_n \end{pmatrix} h = -h^* \begin{pmatrix} ix_1 & & 0 \\ & \ddots & \\ 0 & & ix_n \end{pmatrix} h$$

c) Show that the exponential map of $\mathrm{SO}(2, \mathbb{R})$ is surjective.

Solution. The Lie algebra of $\mathrm{SO}(2, \mathbb{R})$ is $\mathfrak{so}(2, \mathbb{R}) = \mathfrak{o}(2, \mathbb{R}) = \{X \in \mathfrak{gl}(2, \mathbb{R}) : X^t + X = 0\}$. Recall that any element of $\mathrm{SO}(2, \mathbb{R})$ is of the form $\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ for some t. Moreover,

$$\mathrm{Exp} \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Exercise 3 (Abstract Subgroups as Lie Subgroups)**.** Let H be an abstract subgroup of a Lie group G and let $\mathfrak{h}$ be a subspace of the Lie algebra $\mathfrak{g}$ of G. Further let $U \subseteq \mathfrak{g}$ be an open neighborhood of $0 \in \mathfrak{g}$ and let $V \subseteq G$ be an open neighborhood of $e \in G$ such that the exponential map $\exp : U \to V$ is a diffeomorphism satisfying $\exp(U \cap \mathfrak{h}) = V \cap H$. Show that the following statements hold:

a) H is a Lie subgroup of G with the induced relative topology;

b) $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$;

c) $\mathfrak{h}$ is the Lie algebra of H.

Solution. We will first show that H is an embedded submanifold of G. For that it is enough to check that there are slice charts about every point $h \in H$. For $h = e$ choose any linear isomorphism $E : \mathfrak{g} \to \mathbb{R}^m$ that sends $\mathfrak{h}$ to $\mathbb{R}^k$ where $\dim G = \dim \mathfrak{g} = m$ and $\dim \mathfrak{h} = k$. The composite map

$$\varphi = E \circ \exp^{-1} : \exp U = V \longrightarrow \mathbb{R}^m$$

is then a smooth chart for G, and

$$\varphi((\exp(U) \cap H) = E(U \cap \mathfrak{h})$$

is the slice obtained by setting the last $m - k$ coordinates equal to zero. Moreover, if $h \in H$ is arbitrary, the left translation map L_h is a diffeomorphism from $\exp(U)$ to a neighborhood of h. Since H is a subgroup, $L_h(H) = H$, and so

$$L_h((\exp U) \cap H) = L_h(\exp U) \cap H,$$

and $\varphi \circ L_h^{-1}$ is easily seen to be a slice chart for H in a neighborhood of h. Thus H is an embedded submanifold of G.

We will now make use of the following Lemma:

Lemma: Let G be a Lie group, and suppose $H \subseteq G$ is a subgroup that is also an embedded submanifold. Then H is a Lie subgroup.

Proof: We need only check that multiplication $m : H \times H \to H$ and inversion $i : H \to H$ are smooth maps. Because multiplication is a smooth map from $G \times G$ to G its restriction is clearly smooth from $H \times H$ to G. Because H is a subgroup, multiplication takes $H \times H$ to H. Using local slice charts for H in G it follows easily that $m : H \times H \to H$ is smooth. The same argument works for inversion. □

This proves a)). We will prove b) and c) in one go:

Denote by $\iota : H \to G$ the embedding from H into G and let $\mathfrak{b} \subseteq \mathfrak{g}$ be a complementary subspace of $\mathfrak{h}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{b}$. This yields the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Lie}(H) & \xrightarrow{d_e\iota} & \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{b} \\ \big\downarrow \exp & & \big\downarrow \exp \\ H & \xrightarrow{\quad \iota \quad} & G \end{array}$$

By construction of the slice charts of H it is immediate that $d_e\iota$ is an isomorphism of vector spaces from $\mathrm{Lie}(H)$ to $\mathfrak{h}$. Furthermore, ι is a Lie group homomorphism whence its differential $d_e\iota$ induces a Lie algebra homomorphism. Therefore $d_e\iota$ is a Lie algebra isomorphism from $\mathrm{Lie}(H)$ to $\mathfrak{h}$. Under the identification $H \cong \iota(H) \leq G$ we get $\mathrm{Lie}(H) \cong \mathfrak{h}$. This proves b) and c).

Exercise 4 (Lie Group homomorphisms and their differentials)**.** Let G be a connected Lie group, let H be a Lie group and let $\varphi, \psi \colon G \to H$ be Lie group homomorphisms.

Show that $\varphi = \psi$ if and only if $d\varphi = d\psi$.

Solution. If $\varphi = \psi$ then clearly $d\varphi = d\psi$. Thus it suffices to prove the converse direction.

Assume that $d\varphi = d\psi$. We consider the set

$$A \coloneqq \{g \in G \,|\, \varphi(g) = \psi(g)\},$$

and we need to show that $A = G$. Note that A is closed and contains the identity element $e \in A$. Because G is connected we are left to show that A is open.

Let $g_0 \in A$. Recall that there is an open neighborhood $0 \in V \subseteq T_eG \cong \mathfrak{g}$ and an open neighborhood $e \in U \subseteq G$ such that $\exp \colon U \to V$ is a diffeomorphism. Let $g = g_0 v \in g_0 V$ with $v = \exp(X)$ for some $X \in U$. Then

$$\begin{aligned} \varphi(g) &= \varphi(g_0)\varphi(\exp(X)) \\ &= \varphi(g_0)\exp(d\varphi(X)) \\ &= \psi(g_0)\exp(d\psi(X)) \\ &= \psi(g_0)\psi(\exp(X)) = \psi(g), \end{aligned}$$

whence $g_0 V \subseteq A$.

Because g_0 was arbitrary, A is open.

Exercise 5 (Multiplication and exp)**.** Let G be a Lie group with Lie algebra $\mathfrak{g}$. Show that for all $X, Y \in \mathfrak{g}$ and small enough $t \in \mathbb{R}$

$$\exp(tX)\exp(tY) = \exp(t(X+Y) + O(t^2))$$

where $O(t^2)$ is a differentiable $\mathfrak{g}$-valued function such that $\frac{O(t^2)}{t^2}$ is bounded as $t \to 0$.

Solution. Let $X, Y \in \mathfrak{g}$. Let $U \subseteq \mathfrak{g}$ be an open neighborhood about 0 and $V \subseteq G$ be an open neighborhood about $e \in G$ such that $\exp : U \to V$ is a diffeomorphism. Choose $\varepsilon > 0$ such that $\exp(tX)\exp(tY) \in V$ for all $t \in (-\varepsilon, \varepsilon)$.

Because $\exp(tX)\exp(tY) \in V$, for all $|t| < \varepsilon$, and $\exp : U \to V$ is a diffeomorphism, we find a smooth $\mathfrak{g}$-valued function $Z : (-\varepsilon, \varepsilon) \to \mathfrak{g}$ such that

$$\exp(tX)\exp(tY) = \exp(Z(t))$$

for all $|t| < \varepsilon$.

By Taylor's theorem we may write

$$Z(t) = Z(0) + tZ'(0) + O(t^2)$$

where $O(t^2)$ is a smooth $\mathfrak{g}$-valued function such that $\frac{O(t^2)}{t^2}$ is bounded as $t \to 0$. Setting $t = 0$ yields

$$\exp(0) = e = \exp(0 \cdot X)\exp(0 \cdot Y) = \exp(Z(0))$$

and because $\exp : U \to V$ is bijective we have $Z(0) = 0$.

Let $f \in C^\infty(G)$. Then by the chain rule

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} f(\exp(tX)\exp(tY)) &= \left.\frac{d}{dt}\right|_{t=0} f(\exp(tX)\exp(0 \cdot Y)) + \left.\frac{d}{dt}\right|_{t=0} f(\exp(0 \cdot X)\exp(tY)) \\
&= \left.\frac{d}{dt}\right|_{t=0} f(\exp(tX)) + \left.\frac{d}{dt}\right|_{t=0} f(\exp(tY)) \\
&= Xf + Yf,
\end{aligned}$$

and

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} f(\exp(tX)\exp(tY)) &= \left.\frac{d}{dt}\right|_{t=0} f(\exp(Z(t))) \\
&= Z'(0)f
\end{aligned}$$

identifying $\mathfrak{g} \cong T_0\mathfrak{g}$. Therefore $Z'(0) = X + Y \in \mathfrak{g}$ and

$$\exp(tX)\exp(tY) = \exp(Z(t)) = \exp(tZ'(0) + O(t^2)) = \exp(t(X+Y) + O(t^2))$$

for all $|t| < \varepsilon$.