Solutions Exercise Sheet 6

Exercise 1 (Isomorphism theorems for Lie algebras)**.** Let $\mathfrak{g}$ be a Lie algebra.

a) Let $\mathfrak{h} \trianglelefteq \mathfrak{g}$ be an ideal. Show that

$$[X + \mathfrak{h}, Y + \mathfrak{h}] := [X, Y] + \mathfrak{h}$$

defines a Lie algebra structure on $\mathfrak{g}/\mathfrak{h}$.

Solution. All we need to show is that the bracket defined above is well-defined. All the Lie algebra properties will then be inherited from $\mathfrak{g}$. Now let $X, X', Y, Y' \in \mathfrak{g}$ and $U, V \in \mathfrak{h}$ such that $X' = X + U$ and $Y' = Y + V$. Then

$$\begin{aligned}[X' + \mathfrak{h}, Y' + \mathfrak{h}] &= [X + U, Y + V] + \mathfrak{h} \\ &= [X, Y] + \underbrace{[X, V] + [U, Y] + [U, V]}_{\in \mathfrak{h}} + \mathfrak{h} \\ &= [X, Y] + \mathfrak{h}.\end{aligned}$$

This proves that the Lie bracket is well-defined.

b) Show that if $\varphi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism then

$$\mathfrak{g}/\mathrm{ker}\varphi \cong \mathrm{im}\varphi$$

as Lie algebras.

Solution. Let us first see that $\mathrm{ker}\varphi$ is an ideal in $\mathfrak{g}$ whence $\mathfrak{g}/\mathrm{ker}\varphi$ has indeed a Lie algebra structure. Let $X \in \mathrm{ker}\varphi$ and let $Y \in \mathfrak{g}$. Then

$$\varphi([X, Y]) = [\varphi(X), \varphi(Y)] = [0, \varphi(Y)] = 0$$

whence $[X, Y] \in \mathrm{ker}\varphi$. This shows that $\mathrm{ker}\varphi \trianglelefteq \mathfrak{g}$ is an ideal.

Clearly, $\mathrm{im}\varphi \leq \mathfrak{h}$ is a Lie subalgebra. We claim that $\psi : \mathfrak{g}/\mathrm{ker}\varphi \to \mathrm{im}\varphi$ defined by

$$\psi(X + \mathrm{ker}\varphi) = \varphi(X)$$

is a well-defined Lie algebra isomorphism. We have

$$\begin{aligned}X + \mathrm{ker}\varphi = Y + \mathrm{ker}\varphi &\iff X - Y \in \mathrm{ker}\varphi \iff \varphi(X) = \varphi(Y) \\ &\iff \psi(X) = \psi(Y)\end{aligned}$$

for all $X, Y \in \mathfrak{g}$. This proves that ψ is well-defined and injective. Surjectivity is immediate from the definition. Finally, ψ is a Lie algebra homomorphism since

$$\begin{aligned}\psi([X + \ker\varphi, Y + \ker\varphi]) &= \psi([X, Y] + \ker\varphi) = \varphi([X, Y]) \\ &= [\varphi(X), \varphi(Y)] \\ &= [\psi(X + \ker\varphi), \psi(Y + \ker\varphi)]\end{aligned}$$

for all $X, Y \in \mathfrak{g}$.

c) Let $\mathfrak{h} \subseteq \mathfrak{I}$ be ideals of $\mathfrak{g}$. Show that

$$\mathfrak{I}/\mathfrak{h} \trianglelefteq \mathfrak{g}/\mathfrak{h} \quad \text{and} \quad (\mathfrak{g}/\mathfrak{h})/(\mathfrak{I}/\mathfrak{h}) \cong \mathfrak{g}/\mathfrak{I}.$$

Solution. Observe that because $\mathfrak{h} \trianglelefteq \mathfrak{g}$ also $\mathfrak{h} \trianglelefteq \mathfrak{I}$. Consider the homomorphism $\varphi : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{I}$ given by

$$\varphi(X + \mathfrak{h}) = X + \mathfrak{I}.$$

This is a well-defined homomorphism since $\mathfrak{h} \trianglelefteq \mathfrak{I}$. Let $X + \mathfrak{h} \in \ker\varphi$. Then

$$\mathfrak{I} = \varphi(X + \mathfrak{h}) = X + \mathfrak{I} \quad \iff \quad X \in \mathfrak{I},$$

i.e. $\ker\varphi = \mathfrak{I}/\mathfrak{h}$. As we have seen in part a) kernels of Lie algebra homomorphisms are ideals whence $\mathfrak{I}/\mathfrak{h} \trianglelefteq \mathfrak{g}/\mathfrak{h}$ and again by part a)

$$(\mathfrak{g}/\mathfrak{h})/(\mathfrak{I}/\mathfrak{h}) \cong \mathfrak{g}/\mathfrak{I}.$$

d) Let $\mathfrak{h}$ and $\mathfrak{I}$ be ideals of $\mathfrak{g}$. Show that $\mathfrak{h} + \mathfrak{I}$ and $\mathfrak{h} \cap \mathfrak{I}$ are ideals in $\mathfrak{g}$, and that

$$\mathfrak{h}/(\mathfrak{h} \cap \mathfrak{I}) \cong (\mathfrak{h} + \mathfrak{I})/\mathfrak{I}.$$

Solution. Observe that $\mathfrak{I} \trianglelefteq \mathfrak{h} + \mathfrak{I}$ because $\mathfrak{I} \trianglelefteq \mathfrak{g}$. Let $X \in \mathfrak{h}$, $Y \in \mathfrak{I}$ and $Z \in \mathfrak{g}$. Then

$$[Z, X + Y] = [Z, X] + [Z, Y] \in \mathfrak{h} + \mathfrak{I}.$$

This proves that $\mathfrak{h} + \mathfrak{I} \trianglelefteq \mathfrak{g}$ is an ideal

Consider the map $\varphi : \mathfrak{h} \to (\mathfrak{h} + \mathfrak{I})/\mathfrak{I}$ given by

$$\varphi(X) = X + \mathfrak{I}.$$

We have

$$X \in \ker\varphi \iff X + \mathfrak{I} = \mathfrak{I} \iff X \in \mathfrak{I} \cap \mathfrak{h},$$

whence $\ker\varphi = \mathfrak{I} \cap \mathfrak{h}$. Therefore $\mathfrak{I} \cap \mathfrak{h}$ is an ideal in $\mathfrak{h}$.

Finally, φ is surjective: Let $X + Y + \mathfrak{I} \in (\mathfrak{h} + \mathfrak{I})/\mathfrak{I}$. Then

$$X + Y + \mathfrak{I} = X + \mathfrak{I} \in \mathrm{im}\varphi.$$

Exercise 2 (Solvable Lie algebras)**.** a) Show that Lie subalgebras and homomorphic images of solvable Lie algebras are solvable.

Solution. Let $\mathfrak{g}$ be a solvable Lie algebra. Recall that $\mathfrak{g}$ is called solvable if

$$\mathfrak{g} \trianglerighteq \mathfrak{g}^{(1)} \trianglerighteq \cdots \trianglerighteq \mathfrak{g}^{(n)} = 0$$

for some $n \in \mathbb{N}$ where $\mathfrak{g}^{(0)} := \mathfrak{g}$ and inductively

$$\mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] = \mathrm{span}_{\mathbb{R}}\{[X, Y] : X, Y \in \mathfrak{g}^{(i)}\}$$

for every $i \in \mathbb{N}$.

First, let $\mathfrak{h} \leq \mathfrak{g}$ be a Lie subalgebra. Then $\mathfrak{h}^{(0)} = \mathfrak{h} \leq \mathfrak{g} = \mathfrak{g}^{(0)}$ and inductively

$$\mathfrak{h}^{(i+1)} = [\mathfrak{h}^{(i)}, \mathfrak{h}^{(i)}] \subseteq [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] = \mathfrak{g}^{(i+1)}$$

for every $i \in \mathbb{N}$. Hence, if $\mathfrak{g}^{(n)} = 0$ then also $\mathfrak{h}^{(n)} = 0$ and $\mathfrak{h}$ is solvable.

Let $\varphi : \mathfrak{g} \to \mathfrak{a}$ be a Lie algebra homomorphism. We need to see that $\mathrm{im}\varphi \leq \mathfrak{a}$ is solvable. Because φ is a Lie algebra homomorphism we have that

$$(\mathrm{im}\varphi)^{(i)} = \varphi(\mathfrak{g})^{(i)} = \varphi(\mathfrak{g}^{(i)}) \tag{1}$$

for every $i \in \mathbb{N}$. From (**??**) it follows that $(\mathrm{im}\varphi)^{(n)} = \varphi(\mathfrak{g}^{(n)}) = 0$ because $\mathfrak{g}^{(n)} = 0$ for some $n \in \mathbb{N}$ whence $\mathrm{im}\varphi$ is solvable.

b) Show that if $\mathfrak{h}$ and $\mathfrak{I}$ are solvable ideals of a Lie algebra $\mathfrak{g}$ then $\mathfrak{h} + \mathfrak{I}$ is a solvable ideal.

<u>Hint:</u> Use exercise **??**. **??**.

Solution. By **??** we have that

$$(\mathfrak{h} + \mathfrak{I})/\mathfrak{I} \cong \mathfrak{h}/(\mathfrak{h} \cap \mathfrak{I}). \tag{2}$$

Since $\mathfrak{h}$ is solvable so is $\mathfrak{h}/(\mathfrak{h} \cap \mathfrak{I})$ as the image of the quotient homomorphism $\pi : \mathfrak{h} \to \mathfrak{h}/(\mathfrak{h} \cap \mathfrak{I})$. By the isomorphism (**??**) we know that $(\mathfrak{h} + \mathfrak{I})/\mathfrak{I}$ is solvable and

$$0 = ((\mathfrak{h} + \mathfrak{I})/\mathfrak{I})^{(n)} = p(\mathfrak{h} + \mathfrak{I})^{(n)} \overset{(\mathbf{??})}{=} p\left((\mathfrak{h} + \mathfrak{I})^{(n)}\right)$$

for some $n \in \mathbb{N}$, where $p : \mathfrak{h} + \mathfrak{I} \to (\mathfrak{h} + \mathfrak{I})/\mathfrak{I}$ is the quotient homomorphism. Therefore

$$(\mathfrak{h} + \mathfrak{I})^{(n)} \subseteq \ker p = \mathfrak{I}.$$

Because $\mathfrak{I}$ is solvable there is $m \in \mathbb{N}$ such that $\mathfrak{I}^{(m)} = 0$. It follows that

$$(\mathfrak{h} + \mathfrak{I})^{(n+m)} = \left((\mathfrak{h} + \mathfrak{I})^{(n)}\right)^{(m)} \subseteq \mathfrak{I}^{(m)} = 0$$

whence $\mathfrak{h} + \mathfrak{I}$ is solvable.

c) Deduce that every Lie algebra contains a unique maximal solvable ideal.

Solution. Let $\mathfrak{g}$ be a Lie algebra and let

$$\mathfrak{a}_1 \subseteq \ldots \subseteq \mathfrak{a}_k \subseteq \ldots$$

be an increasing sequence of solvable ideals of $\mathfrak{g}$. Note that every $\mathfrak{a}_k$ is a linear subspace of $\mathfrak{a}_{k+1}$ whence the sequence $\{\mathfrak{a}_k\}$ has at most $n = \dim \mathfrak{g}$ different elements. Thus every such sequence has a maximal element and by Zorn's lemma there is a maximal solvable ideal $\mathfrak{s} \trianglelefteq \mathfrak{g}$.

Let $\mathfrak{s}$ and $\mathfrak{s}'$ be two maximal solvable ideals of $\mathfrak{g}$. By part b) $\mathfrak{s} + \mathfrak{s}'$ is also a solvable ideal of $\mathfrak{g}$ and by the maximality of $\mathfrak{s}$ and $\mathfrak{s}'$ we get

$$\mathfrak{s} = \mathfrak{s} + \mathfrak{s}' = \mathfrak{s}'.$$

This proves uniqueness.

The so obtained unique maximal solvable ideal of $\mathfrak{g}$ is called its *radical.*

Exercise 3 (Quotients of Lie groups)**.** Let G be a Lie group and let $K \leq G$ be a closed normal subgroup.

Show that G/K can be equipped with a Lie group structure such that the quotient map $\pi\colon G \to G/K$ is a surjective Lie group homomorphism with kernel K.

Solution. From the lecture we know that there exists a suitable neighborhood $U \subset \mathfrak{g}$ of the origin such that $\exp|_U : U \to \exp(U)$ is a diffeomorphism. Denote by $\mathfrak{k} = \mathrm{Lie}(K)$ the Lie algebra associated to K. Choose any complement $\mathfrak{l}$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{l}$ as vector spaces. Define

$$V := U \cap \mathfrak{l}.$$

Since $V \cap \mathfrak{k} = \{0\}$ it is immediate to verify that $\pi \circ \exp|_V : V \to G/K$ is a homeomorphism onto the image. This gives us a local chart around the point $K \in G/K$. We can get an atlas by suitably translating this chart by the natural action of G on G/K. This gives us back an atlas such that each change of coordinate charts is smooth (since the multiplication in G is smooth).

Note that multiplication and inversion are defined on G/K by passing to the quotient, i.e. the following diagrams commute:

$$\begin{array}{ccc} G \times G & \xrightarrow{\;m\;} & G \\ \big\downarrow{\scriptstyle \pi\times\pi} & & \big\downarrow{\scriptstyle \pi} \\ G/K \times G/K & \dashrightarrow & G/K \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\;i\;} & G \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} \\ G/K & \dashrightarrow & G/K \end{array}$$

By definition, the quotient map $\pi\colon G \to G/K$ is a smooth submersion with respect to this smooth structure. Thus, it follows from the constant rank theorem that multiplication and inversion are smooth, and G/K is a Lie group. Moreover, it is clear from the construction that K is the kernel of π.

For more details see Theorem 21.26 in *John M. Lee, "Intorduction to Smooth Manifolds", Springer (2013)*

Exercise 4 (Common eigenvectors)**.** Let G be a connected Lie group and let $\pi\colon G \to \mathrm{GL}(V)$ be a finite-dimensional complex representation.

A *common eigenvector of* $\{\pi(g) : g \in G\}$ is a vector $v \in V$ such that there is a smooth homomorphism $\chi\colon G \to \mathbb{C}$ with $\pi(g)v = \chi(g) \cdot v$ for all $g \in G$. Similarly, a *common eigenvector of* $\{d_e\pi(X)\colon X \in \mathfrak{g}\}$ is a vector $v \in V$ such that there is a linear functional $\lambda\colon \mathfrak{g} \to \mathbb{C}$ with $d_e\pi(X)v = \lambda(X) \cdot v$ for all $X \in \mathfrak{g}$.

Show that a vector $v \in V$ is a common eigenvector of $\{d_e\pi(X) : X \in \mathfrak{g}\}$ if and only if it is a common eigenvector of $\{\pi(g) : g \in G\}$. Moreover, show that $\chi(\exp(X)) = e^{\lambda(X)}$ for all $X \in \mathfrak{g}$ (with $\chi\colon G \to \mathbb{C}$ and $\lambda\colon \mathfrak{g} \to \mathbb{C}$ as above).

Solution. Let $G_v := \{g \in G : \pi(g)\mathbb{C}v = \mathbb{C}v\}$ be the stabilizer of the line $\mathbb{C}v$. Then G_v is a closed subgroup of G and hence a Lie group whose Lie algebra is

$$\begin{aligned}
\mathrm{Lie}(G_v) &= \{X \in \mathfrak{g} : \exp_G(tX) \in G_v \text{ for all } t \in \mathbb{R}\}\\
&= \{X \in \mathfrak{g} :\ \pi(\exp_G(tX))\mathbb{C}v = \mathbb{C}v \text{ for all } t \in \mathbb{R}\}\\
&= \{X \in \mathfrak{g} :\ \exp_{\mathrm{GL}(V)}(t d_e\pi(X))\mathbb{C}v = \mathbb{C}v \text{ for all } t \in \mathbb{R}\}\,.
\end{aligned}$$

Now observe that if $A \in \mathrm{End}(V)$, then

$$\exp_{\mathrm{GL}(V)}(tA)\mathbb{C}v \Leftrightarrow \mathbb{C}v \Leftrightarrow A(\mathbb{C}v) \subset \mathbb{C}v\,.$$

In fact ($\Leftarrow$) is immediate by the exponential series and ($\Rightarrow$) follows from the fact that $A = \lim_{t\to 0} \frac{\exp_{\mathrm{GL}(V)}(tA) - \mathrm{Id}}{t}$.

Thus

$$\mathrm{Lie}(G_v) = \{X \in \mathfrak{g} :\ d_e\pi(X)(\mathbb{C}v) \subset \mathbb{C}v\} = \mathfrak{g}$$

by hypothesis. Since G s connected, this implies that $G_v = G$. Thus for all $g \in G$ there is a well defined $\chi(g) \in \mathbb{C}^*$ with $\pi(g)v = \chi(g)v$ and since $g \mapsto \pi(g)v$ is smooth, so is χ. Finally,

$$\chi(\exp_G(X))v = \pi(\exp_G(X))v = \exp_{\mathrm{GL}(V)}(d_e\pi(X))v = e^{\lambda(X)}v\,.$$

Exercise 5 (Weight spaces and ideals)**.** Let $\mathfrak{g}$ be a Lie algebra, let $\mathfrak{h} \trianglelefteq \mathfrak{g}$ be an ideal and let $\pi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ a finite-dimensional complex representation. For a given linear functional $\lambda\colon \mathfrak{h} \to \mathbb{C}$ consider its weight space

$$V_\lambda^{\mathfrak{h}} \coloneqq \{v \in V \mid \pi(X)v = \lambda(X)v \quad \forall X \in \mathfrak{h}\}.$$

Show that every weight space $V_\lambda^{\mathfrak{h}}$ is invariant under $\pi(\mathfrak{g})$, i.e. $\pi(Y)V_\lambda^{\mathfrak{h}} \subseteq V_\lambda^{\mathfrak{h}}$ for every $\lambda \in \mathfrak{h}^*, Y \in \mathfrak{g}$.

Solution. Let $\lambda \in \mathfrak{h}^*$, let $Y \in \mathfrak{g}$, let $X \in \mathfrak{h}$ and let $v \in V_\lambda^{\mathfrak{h}}$. Then

$$\begin{aligned}
\pi(X)\pi(Y)v &= (\pi(X)\pi(Y) - \pi(Y)\pi(X))v + \pi(Y)\pi(X)v\\
&= \pi([X,Y])v + \lambda(X)\pi(Y)v\\
&= \lambda([X,Y])v + \lambda(X)\pi(Y)v.
\end{aligned}$$

Thus, we are left to prove that $\lambda([X,Y]) = 0$.

Consider the increasing sequence of subspaces

$$W_m = \langle v, \pi(Y)v, \ldots, \pi(Y)^m v \rangle \leq V, \quad m \geq 0.$$

Because V is finite-dimensional this sequence stabilizes for some $N \in \mathbb{N}$:

$$W_{N-1} \lneq W_N = W_{N+1} = \cdots$$

We claim that for all $m \geq 0$, W_m is invariant under $\pi(\mathfrak{h})$ and furthermore

$$\pi(X)\pi(Y)^m v - \lambda(Y)\pi(X)^m v \in W_{m-1} \qquad \forall X \in \mathfrak{h}. \tag{3}$$

We will prove this by induction on m. It holds for $m = 0$ because $v \in V_\lambda^{\mathfrak{h}}$. So let's assume it holds for $m-1$. We compute:

$$\begin{aligned}\pi(X)\pi(Y)^m v - \lambda(X)\pi(Y)^m v &= [\pi(X),\pi(Y)]\pi(Y)^{m-1}v + \pi(Y)\pi(X)\pi(Y)^{m-1}v - \lambda(X)\pi(Y)^m v \\ &= [\pi(X),\pi(Y)]\pi(Y)^{m-1}v + \pi(Y)\pi(X)\pi(Y)^{m-1}v - \pi(Y)\lambda(X)\pi(Y)^{m-1}v.\end{aligned}$$

By induction hypothesis, we have that

$$w \coloneqq \pi(X)\pi(Y)^{m-1}v - \lambda(Y)\pi(X)^{m-1}v \in W_{m-2},$$

and $\pi(Y)w \in W_{m-1}$ by construction of the W_i'2. Moreover, $\mathfrak{h}$ is an ideal, so that $[\pi(X),\pi(Y)] \in \pi(\mathfrak{h})$ and, by induction hypothesis,

$$[\pi(X),\pi(Y)]\pi(Y)^{m-1}v \in W_{m-1}.$$

Thus,

$$\pi(X)\pi(Y)^m v - \lambda(X)\pi(Y)^m v \in W_{m-1}.$$

We know that W_N is invariant for both $\pi(Y)$ and $\pi(X)$. In particular, (**??**) shows that $\pi(X)$ acts on W_N as an upper triangular matrix in the basis $\{v, \pi(Y)v, \ldots, \pi(Y)^N v\}$:

$$\begin{pmatrix} \lambda(X) & & * \\ & \ddots & \\ 0 & & \lambda(X) \end{pmatrix}$$

Therefore,

$$\mathrm{tr}_{W_N}([\pi(X),\pi(Y)]) = 0 = \mathrm{tr}_{W_N}(\pi([X,Y])) = N\lambda([X,Y]),$$

which implies that $\lambda([X,Y]) = 0$.

Exercise 6 (Lie's theorem for Lie algebras)**.** Let $\mathfrak{g}$ be a solvable Lie algebra and let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional complex representation.

Show that $\rho(\mathfrak{g})$ stabilizes a flag $V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_n = 0$, with $\mathrm{codim} V_i = i$, i.e. $\rho(X)V_i \subseteq V_i$ for every $X \in V_i$, $i = 1, \dots, n$.

<u>Hint:</u> Use exercise **??**.

Solution. By induction, it suffices to show that there is a weight $\lambda \in \mathfrak{g}^*$ for ρ such that $V_\lambda^{\mathfrak{g}} \neq \{0\}$.

We will prove this by induction on $\dim \mathfrak{g}$. The case $\dim \mathfrak{g} = 0$ is trivial. So let's assume that it holds for $\dim \mathfrak{g} = m - 1$.

Since $\mathfrak{g}$ is solvable (of positive dimension) it properly includes $[\mathfrak{g}, \mathfrak{g}]$. Since $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian, any subspace is automatically an ideal. Take a subspace of codimension one in $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. Then its inverse image $\mathfrak{h} \trianglelefteq \mathfrak{g}$ is an ideal of codimension on in $\mathfrak{g}$. Thus, we can decompose

$$\mathfrak{g} = \mathfrak{h} + \mathbb{C}Y$$

for some $Y \in \mathfrak{g}$.

Notice that $\mathfrak{h}$ is a solvable ideal of dimension $m - 1$, whence there is a weight $\lambda \in \mathfrak{h}^*$ such that $V_\lambda^{\mathfrak{h}} \neq \{0\}$. By exercise **??** $V_\lambda^{\mathfrak{h}}$ is invariant under the action of $\rho(\mathfrak{g})$. In particular, $\rho(Y)V_\lambda^{\mathfrak{h}} \subseteq V_\lambda^{\mathfrak{h}}$ and there is $v \in V_\lambda^{\mathfrak{h}} \setminus \{0\}$ such that $\rho(Y)v = \beta v$ for some $\beta \in \mathbb{C}$. We define a linear functional $\lambda' \in \mathfrak{g}^*$ by

$$\lambda'(X + \alpha Y) = \lambda(X) + \alpha\beta$$

for all $X \in \mathfrak{h}, \alpha \in \mathbb{C}$.

By construction $v \in V_{\lambda'}^{\mathfrak{g}} \neq \{0\}$.