Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich
Prof. Dr. Josef Teichmann 13th January 2022

# Functional Analysis I

## FS 2021/2022

## Mock exam

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**Student ID:** (your student ID will be preprinted here) 0

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- Duration: **180 minutes**.
- Place your student ID on your desk.
- Switch off your mobile phone and store it in your bag.
- Start every new exercise on a separate piece of paper and write your student ID on every page you want to hand in. Please leave 2 cm margins on each side of each page for the correction.
- The solutions to exercises 2–4 shall be thoroughly justified.
- Exercise 1 consists of 5 multiple choice (MC) questions. Every question has **exactly one correct answer**. A MC question is awarded 2 points if and only if the correct answer and only the correct answer is clearly marked. In any other case (incorrect answer, no answer, several answers, unclear answer, ...) 0 points are awarded.
- Please use unerasable black or blue pens. **In particular, do not use red or green pens.**

**Please do not fill out the table below!**

| Ex. | 1 | 2 | 3 | 4 | Tot. |
|---|---|---|---|---|---|
| Pts. | [10] | [11] | [11] | [10] | [42] |
| Corr. | | | | | |

| | |
|---|---|
| Completeness | |
| Grade | |

**Exercise 1. Multiple Choice, 10(=2+2+2+2+2) points**

**MC 1. (2 points)** Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed $\mathbb{R}$-vector spaces. $L(X, Y)$ shall denote the space of bounded linear operators mapping from $(X, \|\cdot\|_X)$ to $(Y, \|\cdot\|_Y)$, equipped with the operator norm $\|\cdot\|_{L(X,Y)}$. Which one of the following statements 1) is **true** and 2) is such that it is **not implied by another true statement** out of the four statements?

- ☐ $(L(X, Y), \|\cdot\|_{L(X,Y)})$ is complete if $(X, \|\cdot\|_X)$ is complete.
- ☐ $(L(X, Y), \|\cdot\|_{L(X,Y)})$ is complete if $(Y, \|\cdot\|_Y)$ is complete.
- ☐ $(L(X, Y), \|\cdot\|_{L(X,Y)})$ is complete if both $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are complete.
- ☐ $(L(X, Y), \|\cdot\|_{L(X,Y)})$ is always complete.

**MC 2. (2 points)** Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed $\mathbb{R}$-vector spaces and let $(A_k)_{k\in\mathbb{N}} \subseteq L(X, Y)$ and $A_\infty : X \to Y$ be linear operators (where, for $k \in \mathbb{N}$, $A_k$ is continuous w.r.t. the norm topologies on $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$) such that for every $x \in X$ it holds that $\limsup_{k\to\infty} \|A_k x - A_\infty x\|_Y = 0$. Which one of the following statements 1) is **true** and 2) is such that it is **not implied by another true statement** out of the four statements?

- ☐ $A_\infty$ is continuous if $(X, \|\cdot\|_X)$ is complete.
- ☐ $A_\infty$ is continuous if $(Y, \|\cdot\|_Y)$ is complete.
- ☐ $A_\infty$ is continuous if $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are complete.
- ☐ $A_\infty$ is always continuous.

**MC 3. (2 points)** Which one of the following statements is **false**?

- ☐ $(L^2([0, 1], \mathbb{R}))^*$ is isometrically isomorphic to $L^2([0, 1], \mathbb{R})$.
- ☐ $(L^1([0, 1], \mathbb{R}))^*$ is isometrically isomorphic to $L^\infty([0, 1], \mathbb{R})$.
- ☐ $(L^\infty([0, 1], \mathbb{R}))^*$ is isometrically isomorphic to $L^1([0, 1], \mathbb{R})$.
- ☐ $(L^4([0, 1], \mathbb{R}))^*$ is isometrically isomorphic to $L^{4/3}([0, 1], \mathbb{R})$.

**MC 4. (2 points)** Let $(X, \|\cdot\|_X)$ be a normed $\mathbb{R}$-vector space and let $A, B \subseteq X$ be non-empty disjoint convex sets. In which of the following situations is it assured that there exists $\varphi \in X^*$ such that $\sup_{a\in A} \varphi(a) < \inf_{b\in B} \varphi(b)$?

☐ $A$ open, $B$ closed.

☐ $A$ compact, $B$ open.

☐ $A$ closed, $B$ compact.

☐ $A$ closed, $B$ closed.

**MC 5. (2 points)** Let $c_c := \{(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R} \colon \exists N \in \mathbb{N} \text{ s.t. } x_n = 0 \text{ for all } n > N\}$ denote the space of real-valued sequences with at most finitely many non-zero elements and let $\ell^2 := \{(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R} \colon \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$ denote the ($\mathbb{R}$-Hilbert) space of real-valued square integrable sequences, equipped with the scalar product $\langle\cdot,\cdot\rangle_{\ell^2}$ satisfying for all $x = (x_n)_{n\in\mathbb{N}}, y = (y_n)_{n\in\mathbb{N}} \in \ell^2$ that $\langle x, y\rangle_{\ell^2} = \sum_{n=1}^{\infty} x_n y_n$. Let $A \colon c_c \subseteq \ell^2 \to \ell^2$ be given by $Ax = (nx_n)_{n\in\mathbb{N}}$ for all $x = (x_n)_{n\in\mathbb{N}} \in c_c$. Which one of the following statements is **true**?

☐ $A$ is closed.

☐ $A$ is injective and has closed range.

☐ $A$ is surjective.

☐ $A^*$ is surjective.

**Exercise 2. 11(=1+2+3+2+3) points**

Let $\ell^2$ be defined as

$$\ell^2 := \left\{ (x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R} \colon \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\},$$

equipped as usual with the norm $\|\cdot\|_{\ell^2}$ satisfying for every $x = (x_n)_{n\in\mathbb{N}} \in \ell^2$ that $\|x\|_{\ell^2} := (\sum_{n=1}^{\infty} |x_n|^2)^{1/2}$. For $s \in (0, \infty)$, let $W^s$ be defined as

$$W^s := \left\{ (x_n)_{n\in\mathbb{N}} \in \ell^2 \colon \sum_{n=1}^{\infty} n^{2s} |x_n|^2 < \infty \right\}.$$

**(a) (1 point)** Prove for every $s \in (0, \infty)$ that $W^s$ is a dense subspace of $(\ell^2, \|\cdot\|_{\ell^2})$.

**(b) (2 points)** Prove that $(W^s, \|\cdot\|_{W^s})$ is a Hilbert space where

$$\|x\|_{W^s} := \left( \sum_{n=1}^{\infty} n^{2s} |x_n|^2 \right)^{1/2} \quad \text{for all } x = (x_n)_{n\in\mathbb{N}} \in W^s.$$

**(c) (3 points)** Prove for every $s \in (0, \infty)$ that the embedding $\iota \colon (W^s, \|\cdot\|_{W^s}) \to (\ell^2, \|\cdot\|_{\ell^2})$, defined by $\iota(x) = x$ for every $x \in W^s$, is a compact operator.

**(d) (2 points)** Prove for every $F \in (W^s, \|\cdot\|_{W^s})^*$ that there exists a sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ with $\sum_{n=1}^{\infty} \frac{|f_n|^2}{n^{2s}} < \infty$ such that $F(x) = \sum_{n=1}^{\infty} f_n x_n$ for all $x = (x_n)_{n\in\mathbb{N}} \in W^s$.

**(e) (3 points)** Prove for all $s_1 > s_2 > 0$ that $W^{s_1} \subsetneq W^{s_2}$ and that $W^{s_1}$ is meager in $(W^{s_2}, \|\cdot\|_{W^{s_2}})$.

**Exercise 3. 11(=2+3+2+4) points**

Let $m \in \mathbb{N}$, let $p \in (1,\infty)$, let $\emptyset \neq \Omega \subseteq \mathbb{R}^m$ be a bounded open set, let $k \in L^{\frac{p}{p-1}}(\Omega\times\Omega,\mathbb{R})$, and let $K\colon L^p(\Omega,\mathbb{R}) \to L^{\frac{p}{p-1}}(\Omega,\mathbb{R})$ be defined by

$$(Kf)(x) = \int_\Omega k(x,y)f(y)\,dy \quad \text{for a.e. } x\in\Omega \text{ for all } f\in L^p(\Omega,\mathbb{R}).$$

**(a) (2 points)** Prove that $K$ is a well-defined bounded linear operator with operator norm $\|K\|_{L(L^p(\Omega,\mathbb{R}),L^{p/(p-1)}(\Omega,\mathbb{R}))} \leq \|k\|_{L^{p/(p-1)}(\Omega\times\Omega,\mathbb{R})}$.

**(b) (3 points)** Prove that $K$ is compact.

**(c) (2 points)** Determine the dual operator $K^*$.

**(d) (4 points)** Assume in addition that $1 < p \leq 2$ and show that, for $g \in L^{p/(p-1)}(\Omega,\mathbb{R})$, there exists $f \in L^p(\Omega,\mathbb{R})$ such that

$$f(x) - (Kf)(x) = g(x) \quad \text{for a.e. } x\in\Omega$$

if and only if $\int_\Omega g\varphi\,dx = 0$ for all $\varphi \in L^p(\Omega,\mathbb{R})$ satisfying

$$\varphi(x) = \int_\Omega k(y,x)\varphi(y)\,dy \quad \text{for a.e. } x\in\Omega.$$

**Exercise 4. 10(=2+4+4) points**

Let $m \in \mathbb{N}$, let $p \in (1, \infty)$, let $s \in (0, \infty)$, let $\emptyset \neq \Omega \subseteq \mathbb{R}^m$ be a bounded open set, let $g \in L^{\frac{p}{p-1}}(\mathbb{R}^m, \mathbb{R})$, let $h \in L^p(\Omega, \mathbb{R})$, and let $V \colon L^p(\Omega, \mathbb{R}) \to \mathbb{R}$ and $E \colon L^p(\Omega, \mathbb{R}) \to \mathbb{R}$ be defined by

$$V(f) = \int_\Omega \int_\Omega g(x - y) f(y) f(x) \, dy \, dx \quad \text{for all } f \in L^p(\Omega, \mathbb{R})$$

and

$$E(f) = V(f) + \|f - h\|^s_{L^p(\Omega,\mathbb{R})} \quad \text{for all } f \in L^p(\Omega, \mathbb{R}).$$

**(a) (2 points)** Prove that $V$ is well-defined.

**(b) (4 points)** Prove that $V$ is weakly sequentially continuous.

**(c) (4 points)** Prove that $E|_{\{f \in L^p(\Omega,\mathbb{R}) \colon f \geq 0 \text{ a.e.}\}}$ attains a global minimum under the additional assumption that $g \geq 0$ almost everywhere.