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Functional Analysis I

FS 2021/2022

Mock exam – Solution

Exercise 1. Multiple Choice, 10(=2+2+2+2+2) points

MC 1. (2 points) Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed $\mathbb{R}$-vector spaces. $L(X, Y)$ shall denote the space of bounded linear operators mapping from $(X, \|\cdot\|_X)$ to $(Y, \|\cdot\|_Y)$, equipped with the operator norm $\|\cdot\|_{L(X,Y)}$. Which one of the following statements 1) is **true** and 2) is such that it is **not implied by another true statement** out of the four statements?

☐ $(L(X, Y), \|\cdot\|_{L(X,Y)})$ is complete if $(X, \|\cdot\|_X)$ is complete.

Solution: Let $X = \mathbb{R}$, let $(Y, \|\cdot\|_Y)$ be non-complete (and such spaces exist, e.g., $(c_c, \|\cdot\|_{\ell^\infty})$), let $(y_n)_{n\in\mathbb{N}} \subseteq Y$ be a Cauchy sequence which does not have a limit in $(Y, \|\cdot\|_Y)$, and define, for $n \in \mathbb{N}$, the linear operator $A_n : \mathbb{R} \to Y$ via $A_n(r) = ry_n$ (for all $r \in \mathbb{R}$). Clearly, $(A_n)_{n\in\mathbb{N}} \subseteq L(\mathbb{R}, Y)$ is a Cauchy sequence. But if it had a limit $A_\infty \in L(\mathbb{R}, Y)$ (which would be the case if $(L(\mathbb{R}, Y), \|\cdot\|_{L(\mathbb{R},Y)})$ was complete), then $(y_n)_{n\in\mathbb{N}}$ would converge to $A_\infty(1)$ in Y as $n \to \infty$, contradicting the choice of $(y_n)_{n\in\mathbb{N}}$ as Cauchy sequence without limit.

■ $(L(X, Y), \|\cdot\|_{L(X,Y)})$ is complete if $(Y, \|\cdot\|_Y)$ is complete.

Solution: See Theorem 2.2.4 in M. Struwe's script or exercise 3.1(c).

☐ $(L(X, Y), \|\cdot\|_{L(X,Y)})$ is complete if both $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are complete.

Solution: This statement is true, but it is implied by the second statement (and it is indeed strictly less general than that statement because of the existence of non-complete vector spaces).

☐ $(L(X, Y), \|\cdot\|_{L(X,Y)})$ is always complete.

Solution: This statement is false since the first statement (which would be implied by this one) is also false.

MC 2. (2 points) Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed $\mathbb{R}$-vector spaces and let $(A_k)_{k\in\mathbb{N}} \subseteq L(X,Y)$ and $A_\infty \colon X \to Y$ be linear operators (where, for $k \in \mathbb{N}$, A_k is continuous w.r.t. the norm topologies on $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$) such that for every $x \in X$ it holds that $\limsup_{k\to\infty} \|A_k x - A_\infty x\|_Y = 0$. Which one of the following statements 1) is **true** and 2) is such that it is **not implied by another true statement** out of the four statements?

- ■ A_∞ is continuous if $(X, \|\cdot\|_X)$ is complete.

  Solution: By the Banach–Steinhaus theorem, $\sup_{n\in\mathbb{N}} \|A_n\|_{L(X,Y)} < \infty$ because $\sup_{n\in\mathbb{N}} \|A_n x\|_Y < \infty$ for every $x \in X$ (if $(X, \|\cdot\|_X)$ is complete) and therefore, A_∞ is bounded. See also Theorem 3.1.1 in M. Struwe's script (or the application 'Anwendung 3.1.1' following it).

- □ A_∞ is continuous if $(Y, \|\cdot\|_Y)$ is complete.

  Solution: Let $(X, \|\cdot\|_X) = (c_c, \|\cdot\|_{\ell^\infty})$, $(Y, \|\cdot\|_Y) = (c_0, \|\cdot\|_{\ell^\infty})$ and define, for $k \in \mathbb{N}$, the operator $A_k \colon X \to Y$ as

  $$A_k x = (x_1, 2x_2, 3x_3, \ldots, kx_k, 0, 0, \ldots) \quad \text{for all } x = (x_n)_{n\in\mathbb{N}} \in c_c.$$

  Clearly, for every $k \in \mathbb{N}$, $A_k \in L(X,Y)$ (with $\|A_k\|_{L(X,Y)} = k$). Moreover, for every $x \in c_c$, $(A_k x)_{k\in\mathbb{N}}$ is eventually constant (if $N \in \mathbb{N}$ is such that $x_n = 0$ for all $n > N$, then $A_n x = A_N x$ for all $n > N$) and therefore converging. The limit operator $A_\infty \colon X \to Y$ is given by

  $$A_\infty x = (n x_n)_{n\in\mathbb{N}} \quad \text{for all } x = (x_n)_{n\in\mathbb{N}} \in c_c.$$

  But A_∞ is clearly not bounded.

- □ A_∞ is continuous if $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are complete.

  Solution: This statement is true. But it is implied by the first statement and it is strictly less general than the first statement.

- □ A_∞ is always continuous.

  Solution: This statement is false as the second statement, which would be implied by this one, is already false.

MC 3. (2 points) Which one of the following statements is **false**?

☐ $(L^2([0,1],\mathbb{R}))^*$ is isometrically isomorphic to $L^2([0,1],\mathbb{R})$.

Solution: For every $p \in [1,\infty)$, $(L^p([0,1],\mathbb{R}))^*$ is isometrically isomorphic to $L^{p^*}([0,1],\mathbb{R})$, where $p^* \in [1,\infty]$ is such that $\frac{1}{p}+\frac{1}{p^*}=1$ (with the convention that $\frac{1}{\infty}=0$). For $p=2$, it holds that $p^*=2$, so that $(L^2([0,1],\mathbb{R}))^*$ is isometrically isomorphic to $L^2([0,1],\mathbb{R})$. Alternatively, since $L^2([0,1],\mathbb{R})$ is a real Hilbert space, Riesz's representation theorem for Hilbert spaces implies that $L^2([0,1],\mathbb{R})$ is isometrically isomorphic to its dual.

☐ $(L^1([0,1],\mathbb{R}))^*$ is isometrically isomorphic to $L^\infty([0,1],\mathbb{R})$.

Solution: For every $p \in [1,\infty)$, $(L^p([0,1],\mathbb{R}))^*$ is isometrically isomorphic to $L^{p^*}([0,1],\mathbb{R})$, where $p^* \in [1,\infty]$ is such that $\frac{1}{p}+\frac{1}{p^*}=1$ (with the convention that $\frac{1}{\infty}=0$). For $p=1$, it holds that $p^*=\infty$, so that $(L^1([0,1],\mathbb{R}))^*$ is isometrically isomorphic to $L^\infty([0,1],\mathbb{R})$.

■ $(L^\infty([0,1],\mathbb{R}))^*$ is isometrically isomorphic to $L^1([0,1],\mathbb{R})$.

Solution: If $L^1([0,1],\mathbb{R})$ was isomorphic to $(L^\infty([0,1],\mathbb{R}))^*$, then the dual space of $L^\infty([0,1],\mathbb{R})$ would be separable (since $L^1([0,1],\mathbb{R})$ is separable). This, in turn, would imply that $L^\infty([0,1],\mathbb{R})$ is separable. This is a contradiction (see, e.g., exercise 2.7).

☐ $(L^4([0,1],\mathbb{R}))^*$ is isometrically isomorphic to $L^{4/3}([0,1],\mathbb{R})$.

Solution: For every $p \in [1,\infty)$, $(L^p([0,1],\mathbb{R}))^*$ is isometrically isomorphic to $L^{p^*}([0,1],\mathbb{R})$, where $p^* \in [1,\infty]$ is such that $\frac{1}{p}+\frac{1}{p^*}=1$ (with the convention that $\frac{1}{\infty}=0$). For $p=4$, it holds that $p^*=\frac{4}{3}$, so that $(L^4([0,1],\mathbb{R}))^*$ is isometrically isomorphic to $L^{4/3}([0,1],\mathbb{R})$.

MC 4. (2 points) Let $(X, \|\cdot\|_X)$ be a normed $\mathbb{R}$-vector space and let $A, B \subseteq X$ be non-empty disjoint convex sets. In which of the following situations is it assured that there exists $\varphi \in X^*$ such that $\sup_{a \in A} \varphi(a) < \inf_{b \in B} \varphi(b)$?

- ☐ A open, B closed.

  Solution: Consider, for example, $X = \mathbb{R}$, $A = (-\infty, 0)$ and $B = [0, \infty)$.

- ☐ A compact, B open.

  Solution: Consider, for example, $X = \mathbb{R}$, $A = [-1, 0]$ and $B = (0, \infty)$.

- ■ A closed, B compact.

  Solution: See Theorem 4.5.1 in the script.

- ☐ A closed, B closed.

  Solution: See exercise 7.4 for a counterexample.

MC 5. (2 points) Let $c_c := \{(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}\colon \exists N \in \mathbb{N} \text{ s.t. } x_n = 0 \text{ for all } n > N\}$ denote the space of real-valued sequences with at most finitely many non-zero elements and let $\ell^2 := \{(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}\colon \sum_{n=1}^{\infty}|x_n|^2 < \infty\}$ denote the ($\mathbb{R}$-Hilbert) space of real-valued square integrable sequences, equipped with the scalar product $\langle\cdot,\cdot\rangle_{\ell^2}$ satisfying for all $x = (x_n)_{n\in\mathbb{N}}, y = (y_n)_{n\in\mathbb{N}} \in \ell^2$ that $\langle x, y\rangle_{\ell^2} = \sum_{n=1}^{\infty} x_n y_n$. Let $A\colon c_c \subseteq \ell^2 \to \ell^2$ be given by $Ax = (nx_n)_{n\in\mathbb{N}}$ for all $x = (x_n)_{n\in\mathbb{N}} \in c_c$. Which one of the following statements is **true**?

- ☐ A is closed.

  Solution: A counterexample is given, for example, by the sequence $(x^{(n)})_{n\in\mathbb{N}} \subseteq c_c$, defined via
  $$x_k^{(n)} = \begin{cases} \frac{1}{k^2} & : k \leq n, \\ 0 & : k > n, \end{cases}$$
  for all $k, n \in \mathbb{N}$. Since $x^{(n)} \to (\frac{1}{k^2})_{k\in\mathbb{N}} \in \ell^2$ as $n \to \infty$ and $Ax^{(n)} \to (\frac{1}{k})_{k\in\mathbb{N}} \in \ell^2$ as $n \to \infty$, but $(\frac{1}{k^2})_{k\in\mathbb{N}} \notin c_c$, A is not closed.

- ☐ A is injective and has closed range.

  Solution: $\mathrm{im}(A) = c_c$ is dense, but not closed.

- ☐ A is surjective.

  Solution: $\mathrm{im}(A) = c_c \neq \ell^2$.

- ■ A^* is surjective.

  Solution: Indeed, A^* is given by $D_{A^*} = \{x = (x_n)_{n\in\mathbb{N}} \in \ell^2\colon (nx_n)_{n\in\mathbb{N}} \in \ell^2\}$ and $A^*((x_n)_{n\in\mathbb{N}}) = (nx_n)_{n\in\mathbb{N}}$ for every $(x_n)_{n\in\mathbb{N}} \in D_{A^*}$. This operator is clearly surjective.

Exercise 2. 11(=1+2+3+2+3) points

Let ℓ^2 be defined as

$$\ell^2 := \left\{ (x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R} \colon \sum_{n=1}^{\infty} |x_n|^2 < \infty \right\},$$

equipped as usual with the norm $\|\cdot\|_{\ell^2}$ satisfying for every $x = (x_n)_{n\in\mathbb{N}} \in \ell^2$ that $\|x\|_{\ell^2} := (\sum_{n=1}^{\infty} |x_n|^2)^{1/2}$. For $s \in (0,\infty)$, let W^s be defined as

$$W^s := \left\{ (x_n)_{n\in\mathbb{N}} \in \ell^2 \colon \sum_{n=1}^{\infty} n^{2s} |x_n|^2 < \infty \right\}.$$

(a) (1 point) Prove for every $s \in (0,\infty)$ that W^s is a dense subspace of $(\ell^2, \|\cdot\|_{\ell^2})$.

Solution: Let $c_c := \{(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R} \colon \exists N \in \mathbb{N} \colon x_n = 0 \text{ for all } n > N\}$. Clearly, $c_c \subseteq W^s$ for every $s \in (0,\infty)$. Since c_c lies dense in $(\ell^2, \|\cdot\|_{\ell^2})$, W^s is a dense subset of $(\ell^2, \|\cdot\|_{\ell^2})$. Moreover, for all $s \in (0,\infty)$, $x = (x_n)_{n\in\mathbb{N}}, y = (y_n)_{n\in\mathbb{N}} \in W^s$, $\alpha \in \mathbb{R}$, it holds that

$$\sum_{n=1}^{\infty} n^{2s} |\alpha x_n + y_n|^2 \leq 2|\alpha|^2 \sum_{n=1}^{\infty} n^{2s} |x_n|^2 + 2 \sum_{n=1}^{\infty} n^{2s} |y_n|^2 < \infty,$$

i.e., $\alpha x + y \in W^s$.

(b) (2 points) Prove that $(W^s, \|\cdot\|_{W^s})$ is a Hilbert space where

$$\|x\|_{W^s} := \left(\sum_{n=1}^{\infty} n^{2s} |x_n|^2 \right)^{1/2} \quad \text{for all } x = (x_n)_{n\in\mathbb{N}} \in W^s.$$

Solution: From (a), we know that W^s is an $\mathbb{R}$-vector space. Define $\langle \cdot,\cdot \rangle_{W^s} \colon W^s \times W^s \to \mathbb{R}$ via

$$\langle x, y \rangle_{W^s} := \sum_{n=1}^{\infty} n^{2s} x_n y_n \quad \text{for all } x = (x_n)_{n\in\mathbb{N}}, y = (y_n)_{n\in\mathbb{N}} \in W^s.$$

Clearly, for all $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in W^s$, it holds by Hölder's inequality (or Cauchy–Schwarz) that

$$\sum_{n=1}^{\infty} n^{2s} |x_n||y_n| \leq \left(\sum_{n=1}^{\infty} n^{2s} |x_n|^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} n^{2s} |y_n|^2 \right)^{1/2} < \infty,$$

so that $\langle \cdot,\cdot \rangle_{W^s} \colon W^s \times W^s \to \mathbb{R}$ is well-defined. Clearly, $\langle \cdot,\cdot \rangle_{W^s} \colon W^s \times W^s \to \mathbb{R}$ is bilinear, $\langle x, x \rangle_{W^s} \geq 0$ for every $x \in W^s$ and $\|x\|_{W^s} = \sqrt{\langle x, x \rangle_{W^s}}$ for every $x \in W^s$. Finally, if $\langle x, x \rangle_{W^s} = 0$ for $x = (x_n)_{n\in\mathbb{N}} \in W^s$, then $n^{2s}|x_n|^2 = 0$ for all $n \in \mathbb{N}$, i.e.,

$x_n = 0$ for all $n \in \mathbb{N}$. Thus, $\langle\cdot,\cdot\rangle_{W^s}\colon W^s \times W^s \to \mathbb{R}$ is a scalar product on W^s. It remains to show that $(W^s, \|\cdot\|_{W^s})$ is complete. For this, let $(x^{(n)})_{n\in\mathbb{N}} \subseteq W^s$ (where $x^{(n)} = (x_k^{(n)})_{k\in\mathbb{N}}$ for every $n \in \mathbb{N}$) be Cauchy w.r.t. $\|\cdot\|_{W^s}$. This means nothing else but that $(y^{(n)})_{n\in\mathbb{N}} \subseteq \ell^2$, given by $y_k^{(n)} = k^s x_k^{(n)}$ for all $k, n \in \mathbb{N}$, is a Cauchy sequence in $(\ell^2, \|\cdot\|_{\ell^2})$. As $(\ell^2, \|\cdot\|_{\ell^2})$ is complete, there exists $y^{(\infty)} = (y_k^{(\infty)})_{k\in\mathbb{N}} \in \ell^2$ so that $\limsup_{n\to\infty}\|y^{(n)} - y^{(\infty)}\|_{\ell^2} = 0$. Clearly, the sequence $x^{(\infty)} = (x_k^{(\infty)})_{k\in\mathbb{N}} \subseteq \mathbb{R}$, defined by $x_k^{(\infty)} = k^{-s} y_k^{(\infty)}$ for every $k \in \mathbb{N}$, belongs to ℓ^2 (since $s > 0$ and $y^{(\infty)} \in \ell^2$) and to W^s (since $y^{(\infty)} \in \ell^2$) and

$$\limsup_{n\to\infty} \|x^{(n)} - x^{(\infty)}\|_{W^s} = \limsup_{n\to\infty} \|y^{(n)} - y^{(\infty)}\|_{\ell^2} = 0.$$

This shows that $(W^s, \|\cdot\|_{W^s})$ is complete. Since $\|\cdot\|_{W^s}$ is induced by the scalar product $\langle\cdot,\cdot\rangle_{W^s}$, it is even a Hilbert space.

Alternatively, consider the bounded linear operator $A\colon \ell^2 \to \ell^2$, defined by $Ax = (n^{-s}x_n)_{n\in\mathbb{N}}$ for $x = (x_n)_{n\in\mathbb{N}} \in \ell^2$. Then, clearly, A is a bijection onto its image and $W^s = \operatorname{im}(A)$. Moreover, equipping $W^s = \operatorname{im}(A)$ with the norm $\|\cdot\|_{W^s}$ turns A into an isometry between $(\ell^2, \|\cdot\|_{\ell^2})$ and $(W^s, \|\cdot\|_{W^s})$. Hence, $(W^s, \|\cdot\|_{W^s})$ is a Hilbert space and the scalar product is given by

$$\langle x, y\rangle_{W^s} = \langle A^{-1}x, A^{-1}y\rangle_{\ell^2} = \sum_{n=1}^{\infty} n^{2s} x_n y_n \quad \text{for all } x = (x_n)_{n\in\mathbb{N}}, y = (y_n)_{n\in\mathbb{N}} \in W^s.$$

(c) (3 points) Prove for every $s \in (0,\infty)$ that the embedding $\iota\colon (W^s, \|\cdot\|_{W^s}) \to (\ell^2, \|\cdot\|_{\ell^2})$, defined by $\iota(x) = x$ for every $x \in W^s$, is a compact operator.

Solution: <u>Way 1)</u> Let $(x^{(n)})_{n\in\mathbb{N}} \subseteq W^s$ be a sequence with $\|x^{(n)}\|_{W^s} \leq 1$ for all $n \in \mathbb{N}$, i.e.,

$$\sum_{k=1}^{\infty} |k|^{2s} |x_k^{(n)}|^2 \leq 1 \quad \text{for all } n \in \mathbb{N}. \tag{1}$$

On one hand, this implies that $\sup_{n\in\mathbb{N}}|x_k^{(n)}| \leq \frac{1}{k^s} \leq 1$ for all $k \in \mathbb{N}$. Passing to a subsequence (by subsequently choosing subsequences and then using the usual diagonal trick), if necessary, we may assume that there exists $x^{(\infty)} = (x_k^{(\infty)})_{k\in\mathbb{N}} \subseteq \mathbb{R}$ such that

$$\limsup_{n\to\infty} |x_k^{(n)} - x_k^{(\infty)}| = 0 \quad \text{for all } k \in \mathbb{N}.$$

Combining this with (1), we obtain in addition that

$$\begin{aligned}\sum_{k=1}^{\infty} k^{2s}|x_k^{(\infty)}|^2 &= \sup_{N\in\mathbb{N}} \left(\sum_{k=1}^{N} k^{2s}|x_k^{(\infty)}|^2\right) \leq \sup_{N\in\mathbb{N}}\sup_{n\in\mathbb{N}} \left(\sum_{k=1}^{N} k^{2s}|x_k^{(n)}|^2\right) \\ &\leq \sup_{n\in\mathbb{N}} \left(\sum_{k=1}^{\infty} k^{2s}|x_k^{(n)}|^2\right) \leq 1.\end{aligned} \tag{2}$$

Moreover, (1) implies in particular for all $n, N \in \mathbb{N}$ that

$$N^{2s} \sum_{k=N}^{\infty} |x_k^{(n)}|^2 \leq \sum_{k=N}^{\infty} k^{2s} |x_k^{(n)}|^2 \leq 1$$

(and (2) implies that $\sum_{k=N}^{\infty} |x_k^{(\infty)}|^2 \leq \frac{1}{N^{2s}}$ for all $N \in \mathbb{N}$). This implies for all $N \in \mathbb{N}$ that

$$\begin{aligned}\limsup_{n\to\infty} \left(\sum_{k=1}^{\infty} |x_k^{(n)} - x_k^{(\infty)}|^2 \right) &\leq \underbrace{\limsup_{n\to\infty} \left(\sum_{k=1}^{N} |x_k^{(n)} - x_k^{(\infty)}|^2 \right)}_{=0} + \limsup_{n\to\infty} \left(\sum_{k>N} |x_k^{(n)} - x_k^{(\infty)}|^2 \right) \\ &\leq 2 \limsup_{n\to\infty} \left(\sum_{k>N} |x_k^{(n)}|^2 \right) + 2 \limsup_{n\to\infty} \left(\sum_{k>N} |x_k^{(\infty)}|^2 \right) \leq \frac{4}{N^{2s}},\end{aligned}$$

which, in turn, proves that $\limsup_{n\to\infty} \|x^{(n)} - x^{(\infty)}\|_{\ell^2} = 0$. Thus, we demonstrated that ι is a compact operator by showing that bounded sets in $(W^s, \|\cdot\|_{W^s})$ are relatively compact when considered as subsets of $(\ell^2, \|\cdot\|_{\ell^2})$.

Way 2) Define, for every $n \in \mathbb{N}$, the operator $I_n \colon (W^s, \|\cdot\|_{W^s}) \to (\ell^2, \|\cdot\|_{\ell^2})$ by

$$I_n(x) = (x_1, x_2, \ldots, x_n, 0, 0, \ldots) \quad \text{for every } x = (x_k)_{k\in\mathbb{N}} \in (W^s, \|\cdot\|_{W^s}).$$

It clearly holds for every $n \in \mathbb{N}$ that $I_n \in L((W^s, \|\cdot\|_{W^s}), (\ell^2, \|\cdot\|_{\ell^2}))$. Moreover, since I_n has finite-dimensional range for every $n \in \mathbb{N}$, the operators I_n are all compact operators. In addition, it holds for every $x = (x_k)_{k\in\mathbb{N}} \in W^s$ that

$$\|I_n x - \iota x\|_{\ell^2}^2 = \sum_{k=n+1}^{\infty} |x_k|^2 \leq \frac{1}{n^{2s}} \sum_{k=n+1}^{\infty} k^{2s} |x_k|^2 \leq \frac{1}{n^{2s}} \|x\|_{W^s}^2.$$

This implies that

$$\limsup_{n\to\infty} \|I_n - \iota\|_{L((W^s, \|\cdot\|_{W^s}), (\ell^2, \|\cdot\|_{\ell^2}))} = 0.$$

Since the space of compact operators $K((W^s, \|\cdot\|_{W^s}), (\ell^2, \|\cdot\|_{\ell^2}))$ is a closed subspace of $L((W^s, \|\cdot\|_{W^s}), (\ell^2, \|\cdot\|_{\ell^2}))$ (equipped with the operator norm), the limit ι is also compact.

Way 3) Let $(x^{(n)})_{n\in\mathbb{N}} \subseteq W^s$ be a weakly converging sequence in $(W^s, \|\cdot\|_{W^s})$ with weak limit $x^{(\infty)} \in W^s$. This implies, in particular, for every $k \in \mathbb{N}$ that

$$\limsup_{n\to\infty} |k^{2s} x_k^{(n)} - k^{2s} x_k^{(\infty)}| = 0.$$

Moreover, since $(x^{(n)})_{n\in\mathbb{N}}$ is weakly converging, it holds (by the Banach–Steinhaus theorem) that

$$C := \sup_{n\in\mathbb{N}} \left(\sum_{k=1}^{\infty} k^{2s} |x_k^{(n)}|^2 \right) < \infty.$$

The convergence of $(x_k^{(n)})_{n\in\mathbb{N}}$ to $x_k^{(\infty)}$ in $\mathbb{R}$ as $n \to \infty$ (for every $k \in \mathbb{N}$) also implies that $\sum_{k=1}^{\infty} k^{2s} |x_k^{(\infty)}|^2 \le C < \infty$. The rest of the argument can now be carried out similar to Way 1), that is, noting that for all $N \in \mathbb{N}$ it holds that

$$\sup_{n\in\mathbb{N}} \left(\sum_{k=N}^{\infty} |x_k^{(n)}|^2 \right) \le \frac{C}{N^{2s}} \quad \text{and} \quad \sum_{k=N}^{\infty} |x_k^{(\infty)}|^2 \le \frac{C}{N^{2s}}$$

we estimate for all $N \in \mathbb{N}$ that

$$\begin{aligned} \limsup_{n\to\infty} \left(\sum_{k=1}^{\infty} |x_k^{(n)} - x_k^{(\infty)}|^2 \right) &\le \underbrace{\limsup_{n\to\infty} \left(\sum_{k=1}^{N} |x_k^{(n)} - x_k^{(\infty)}|^2 \right)}_{=0} + \limsup_{n\to\infty} \left(\sum_{k>N} |x_k^{(n)} - x_k^{(\infty)}|^2 \right) \\ &\le 2 \limsup_{n\to\infty} \left(\sum_{k>N} |x_k^{(n)}|^2 \right) + 2 \limsup_{n\to\infty} \left(\sum_{k>N} |x_k^{(\infty)}|^2 \right) \le \frac{4C}{N^{2s}}, \end{aligned}$$

which implies that $\limsup_{n\to\infty} \|x^{(n)} - x^{(\infty)}\|_{\ell^2} = 0$.

(d) (2 points) Prove for every $F \in (W^s, \|\cdot\|_{W^s})^*$ that there exists a sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ with $\sum_{n=1}^{\infty} \frac{|f_n|^2}{n^{2s}} < \infty$ such that $F(x) = \sum_{n=1}^{\infty} f_n x_n$ for all $x = (x_n)_{n\in\mathbb{N}} \in W^s$.

Solution: Let $F \in (W^s, \|\cdot\|_{W^s})^*$ be arbitrary. Since $(W^s, \|\cdot\|_{W^s})$ is a real Hilbert space according to (b), it holds by Riesz's representation theorem for Hilbert spaces that there exists $\varphi = (\varphi_n)_{n\in\mathbb{N}} \in W^s$ such that

$$F(x) = \langle \varphi, x \rangle_{W^s} = \sum_{n=1}^{\infty} n^{2s} \varphi_n x_n \quad \text{for every } x = (x_n)_{n\in\mathbb{N}} \in W^s.$$

Defining $f_n := n^{2s} \varphi_n$ for every $n \in \mathbb{N}$, we obtain that

$$F(x) = \sum_{n=1}^{\infty} f_n x_n \quad \text{for every } x = (x_n)_{n\in\mathbb{N}} \in W^s$$

and

$$\sum_{n=1}^{\infty} \frac{|f_n|^2}{n^{2s}} = \sum_{n=1}^{\infty} n^{2s} |\varphi_n|^2 = \|\varphi\|_{W^s}^2 < \infty.$$

(e) (3 points) Prove for all $s_1 > s_2 > 0$ that $W^{s_1} \subsetneq W^{s_2}$ and that W^{s_1} is meager in $(W^{s_2}, \|\cdot\|_{W^{s_2}})$.

Solution: Let $s_1, s_2 \in (0, \infty)$ with $s_1 > s_2$. For all $x = (x_n)_{n\in\mathbb{N}} \in W^{s_1}$ it holds that

$$\sum_{n=1}^{\infty} n^{2s_2}|x_n|^2 = \sum_{n=1}^{\infty} \underbrace{n^{2(s_2-s_1)}}_{\leq 1} n^{2s_1}|x_n|^2 \leq \sum_{n=1}^{\infty} n^{2s_1}|x_n|^2 = \|x\|_{W^{s_1}}^2 < \infty$$

(since $s_2 < s_1$). Hence, $W^{s_1} \subseteq W^{s_2}$ (and the embedding is continuous).

Setting $\alpha := \frac{s_1+s_2+1}{2}$, we observe that the sequence $z = (z_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$, given by $z_n = n^{-\alpha}$ for every $n \in \mathbb{N}$, satisfies that

- $z \in W^{s_2}$ as
$$\sum_{n=1}^{\infty}|z_n|^2 \leq \sum_{n=1}^{\infty} n^{2s_2}|z_n|^2 = \sum_{n=1}^{\infty} n^{2s_2-2\alpha} = \sum_{n=1}^{\infty} n^{2s_2-s_1-s_2-1} = \sum_{n=1}^{\infty} n^{s_2-s_1-1} < \infty$$
(since $s_2 > 0$ and $s_2 - s_1 - 1 < -1$) and
- $z \notin W^{s_1}$ as
$$\sum_{n=1}^{\infty} n^{2s_1}|z_n|^2 = \sum_{n=1}^{\infty} n^{2s_1-2\alpha} = \sum_{n=1}^{\infty} n^{2s_1-s_1-s_2-1} = \sum_{n=1}^{\infty} n^{s_1-s_2-1} = \infty$$
(since $s_1 - s_2 - 1 > -1$),

so that $z \in W^{s_2} \setminus W^{s_1}$.

Finally, to prove that W^{s_1} is a meagre subset of $(W^{s_2}, \|\cdot\|_{W^{s_2}})$, consider the sets A_n, $n \in \mathbb{N}$, given by

$$A_n := \left\{ x = (x_k)_{k\in\mathbb{N}} \in W^{s_2} \colon \sum_{k=1}^{\infty} k^{2s_1}|x_k|^2 \leq n^2 \right\}.$$

Clearly, $\bigcup_{n\in\mathbb{N}} A_n = W^{s_1} \cap W^{s_2} = W^{s_1}$. Moreover, it holds for every $n \in \mathbb{N}$ that A_n is closed. Indeed, if $n \in \mathbb{N}$ and $(x^{(k)})_{k\in\mathbb{N}} \subseteq A_n$ converges to $x^{(\infty)}$ in $(W^{s_2}, \|\cdot\|_{W^{s_2}})$ as $k \to \infty$, then, clearly, $x_l^{(k)} \to x_l^{(\infty)}$ for every $l \in \mathbb{N}$ as $k \to \infty$ and therefore, for every $N \in \mathbb{N}$:

$$\sum_{l=1}^{N} l^{2s_1}|x_l^{(\infty)}|^2 = \lim_{k\to\infty} \left(\sum_{l=1}^{N} l^{2s_1}|x_l^{(k)}|^2 \right) \leq \limsup_{k\to\infty} \|x^{(k)}\|_{W^{s_1}}^2 \leq n^2,$$

which implies $x^{(\infty)} \in A_n$. It remains to show for every $n \in \mathbb{N}$ that the interior of A_n is empty. For this, let $n \in \mathbb{N}$ and $y \in A_n$ be arbitrary and let $x \in W^{s_2} \setminus W^{s_1}$. Then it holds for every $k \in \mathbb{N}$ that $y + \frac{1}{k}x \in W^{s_2} \setminus W^{s_1} \subseteq W^{s_2} \setminus A_n$. Moreover, it holds that $y + \frac{1}{k}x \to y$ in $(W^{s_2}, \|\cdot\|_{W^{s_2}})$ as $k \to \infty$.

Exercise 3. 11(=2+3+2+4) points

Let $m \in \mathbb{N}$, let $p \in (1,\infty)$, let $\emptyset \neq \Omega \subseteq \mathbb{R}^m$ be a bounded open set, let $k \in L^{\frac{p}{p-1}}(\Omega \times \Omega, \mathbb{R})$, and let $K\colon L^p(\Omega,\mathbb{R}) \to L^{\frac{p}{p-1}}(\Omega,\mathbb{R})$ be defined by

$$(Kf)(x) = \int_\Omega k(x,y)f(y)\,dy \quad \text{for a.e. } x \in \Omega \text{ for all } f \in L^p(\Omega,\mathbb{R}).$$

(a) (2 points) Prove that K is a well-defined bounded linear operator with operator norm $\|K\|_{L(L^p(\Omega,\mathbb{R}),L^{p/(p-1)}(\Omega,\mathbb{R}))} \leq \|k\|_{L^{p/(p-1)}(\Omega\times\Omega,\mathbb{R})}$.

Solution: We only argue for well-definedness and boundedness as linearity is clear. By Hölder's inequality, it holds for every $f \in L^p(\Omega,\mathbb{R})$ that

$$\begin{aligned}\int_\Omega \left|\int_\Omega |k(x,y)f(y)|\,dy\right|^{\frac{p}{p-1}}\,dx &\leq \int_\Omega\int_\Omega |k(x,y)|^{\frac{p}{p-1}}\,dy\,\|f\|^{\frac{p}{p-1}}_{L^p(\Omega,\mathbb{R})}\,dx \\ &= \|k\|^{\frac{p}{p-1}}_{L^{p/(p-1)}(\Omega\times\Omega,\mathbb{R})}\|f\|^{\frac{p}{p-1}}_{L^p(\Omega,\mathbb{R})}.\end{aligned}$$

The Fubini–Tonelli theorem hence implies that $Kf \in L^{p/(p-1)}(\Omega,\mathbb{R})$ for every $f \in L^p(\Omega,\mathbb{R})$. Moreover, the above computation demonstrates that $\|K\|_{L(L^p(\Omega,\mathbb{R}),L^{p/(p-1)}(\Omega,\mathbb{R}))} \leq \|k\|_{L^{p/(p-1)}(\Omega\times\Omega,\mathbb{R})}$.

(b) (3 points) Prove that K is compact.

Solution: Since $L^p(\Omega,\mathbb{R})$ is reflexive (as $p \in (1,\infty)$), for proving compactness of K it is sufficient to prove that K maps weakly converging sequences to strongly converging sequences as every bounded sequence in $L^p(\Omega,\mathbb{R})$ possesses a weakly converging subsequence.

Now, let $(f_n)_{n\in\mathbb{N}} \subseteq L^p(\Omega,\mathbb{R})$ and $f_\infty \in L^p(\Omega,\mathbb{R})$ be such that $f_n \xrightarrow{\text{w}} f_\infty$ in $L^p(\Omega,\mathbb{R})$ as $n \to \infty$. We are going to prove that $Kf_n \to Kf_\infty$ strongly in $L^{p/(p-1)}(\Omega,\mathbb{R})$ as $n \to \infty$. Note that, since for a.e. $x \in \Omega$, $k(x,\cdot) \in L^{p/(p-1)}(\Omega,\mathbb{R})$, and since $f_n \xrightarrow{\text{w}} f_\infty$ in $L^p(\Omega,\mathbb{R})$ as $n \to \infty$, it holds for a.e. $x \in \Omega$ that

$$\begin{aligned}&\limsup_{n\to\infty}|(Kf_n)(x) - (Kf_\infty)(x)| \\ &= \limsup_{n\to\infty}\left|\int_\Omega k(x,y)f_n(y)\,dy - \int_\Omega k(x,y)f_\infty(y)\,dy\right| = 0.\end{aligned} \tag{3}$$

Moreover, since $(f_n)_{n\in\mathbb{N}} \subseteq L^p(\Omega,\mathbb{R})$ is bounded (by the uniform boundedness principle) and since $\|f_\infty\|_{L^p(\Omega,\mathbb{R})} \leq \sup_{n\in\mathbb{N}}\|f_n\|_{L^p(\Omega,\mathbb{R})}$ (by the fact that $f_\infty \in \overline{\operatorname{conv}}(f_n\colon n \in \mathbb{N})$ or Mazur's lemma or the weak lower semicontinuity of the norm), we obtain for almost every $x \in \Omega$ that

$$\begin{aligned}\sup_{n\in\mathbb{N}}|(Kf_n)(x) - (Kf_\infty)(x)| &= \sup_{n\in\mathbb{N}}\left|\int_\Omega k(x,y)(f_n(y) - f_\infty(y))\,dy\right| \\ &\leq 2\left[\int_\Omega |k(x,y)|^{\frac{p}{p-1}}\,dy\right]^{\frac{p-1}{p}}\sup_{n\in\mathbb{N}}\|f_n\|_{L^p(\Omega,\mathbb{R})},\end{aligned} \tag{4}$$

where the function (in x) on the right hand side (more precisely, the equivalence class of functions) belongs to $L^{p/(p-1)}(\Omega,\mathbb{R})$. By (3) and (4), Lebesgue's dominated convergence theorem implies that

$$\begin{aligned}&\limsup_{n\to\infty}\|Kf_n-Kf_\infty\|_{L^{p/(p-1)}(\Omega,\mathbb{R})}\\&=\limsup_{n\to\infty}\left[\int_\Omega\left|\int_\Omega k(x,y)(f_n(y)-f_\infty(y))\,dy\right|^{\frac{p}{p-1}}\,dx\right]^{\frac{p-1}{p}}=0,\end{aligned}$$

i.e., K is compact.

(c) (2 points) Determine the dual operator K^*.

Solution: For every $q\in(1,\infty)$, let $J_q\colon L^{q/(q-1)}(\Omega,\mathbb{R})\to(L^q(\Omega,\mathbb{R}))^*$ be given by

$$(J_qg)(f)=\int_\Omega gf\,dx\quad\text{for all } f\in L^q(\Omega,\mathbb{R}),g\in L^{q/(q-1)}(\Omega,\mathbb{R}).$$

Riesz's representation theorem ensures for every $q\in(1,\infty)$ that J_q is an isometric isomorphism. Thus, we obtain – using Fubini's theorem – for all $f,g\in L^p(\Omega,\mathbb{R})$ and $\varphi:=J_{p/(p-1)}g\in(L^{p/(p-1)}(\Omega,\mathbb{R}))^*$ that

$$\begin{aligned}\varphi(Kf)=(J_{p/(p-1)}g)(Kf)&=\int_\Omega g(x)(Kf)(x)\,dx\\&=\int_\Omega g(x)\int_\Omega k(x,y)f(y)\,dy\,dx\\&=\int_\Omega\int_\Omega k(y,x)g(y)\,dy\,f(x)\,dx\\&=\int_\Omega(\tilde{K}g)(x)f(x)\,dx,\end{aligned}$$

where $\tilde{K}\colon L^p(\Omega,\mathbb{R})\to L^{p/(p-1)}(\Omega,\mathbb{R})$ shall be defined as

$$(\tilde{K}h)(x)=\int_\Omega k(y,x)h(y)\,dy\quad\text{for a.e. } x\in\Omega\text{ for all } h\in L^p(\Omega,\mathbb{R}).$$

(Well-definedness, boundedness and linearity of $\tilde{K}$ are clear by (a).) Continuing the previous calculation, we obtain for all $f,g\in L^p(\Omega,\mathbb{R})$ and $\varphi:=J_{p/(p-1)}g\in(L^{p/(p-1)}(\Omega,\mathbb{R}))^*$ that

$$(K^*\varphi)(f)=\varphi(Kf)=(J_p\tilde{K}g)(f)=(J_p\tilde{K}J_{p/(p-1)}^{-1}\varphi)(f),$$

which implies that $K^*=J_p\tilde{K}J_{p/(p-1)}^{-1}$. Or, sloppily speaking, identifying $(L^p(\Omega,\mathbb{R}))^*$ with $L^{p/(p-1)}(\Omega,\mathbb{R})$ and $(L^{p/(p-1)}(\Omega,\mathbb{R}))$ with $L^p(\Omega,\mathbb{R})$ (via J_p and $J_{p/(p-1)}$, respectively) K^* can be considered as a linear operator from $L^p(\Omega,\mathbb{R})$ to $L^{p/(p-1)}(\Omega,\mathbb{R})$ and, as such, coincides with $\tilde{K}$ defined above since $\int_\Omega(Kf)(x)g(x)\,dx=\int_\Omega f(x)(\tilde{K}g)(x)\,dx$ for all $f,g\in L^p(\Omega,\mathbb{R})$.

(d) (4 points) Assume in addition that $1 < p \leq 2$ and show that, for $g \in L^{p/(p-1)}(\Omega, \mathbb{R})$, there exists $f \in L^p(\Omega, \mathbb{R})$ such that

$$f(x) - (Kf)(x) = g(x) \quad \text{for a.e. } x \in \Omega$$

if and only if $\int_\Omega g\varphi \, dx = 0$ for all $\varphi \in L^p(\Omega, \mathbb{R})$ satisfying

$$\varphi(x) = \int_\Omega k(y,x)\varphi(y)\, dy \quad \text{for a.e. } x \in \Omega.$$

Solution: First, note that if $g \in L^{p/(p-1)}(\Omega, \mathbb{R})$ satisfies $g(x) = f(x) - (Kf)(x)$ for a.e. $x \in \Omega$ for some $f \in L^p(\Omega, \mathbb{R})$, then – since $Kf \in L^{p/(p-1)}(\Omega, \mathbb{R})$ – f has also to belong to $L^{p/(p-1)}(\Omega, \mathbb{R})$. That is, it is enough to consider the restriction of K to $L^{p/(p-1)}(\Omega, \mathbb{R})$. Since $p \in (1, 2]$, we obtain that $\frac{p}{p-1} \geq 2 \geq p$ so that, as Ω is bounded, $L^{p/(p-1)}(\Omega, \mathbb{R})$ embeds continuously into $L^p(\Omega, \mathbb{R})$. Letting $\iota \colon L^{p/(p-1)}(\Omega, \mathbb{R}) \to L^p(\Omega, \mathbb{R})$ denote the canonical embedding, we obtain from the above considerations that $g \in L^{p/(p-1)}(\Omega, \mathbb{R})$ satisfies that $g(x) = f(x) - (Kf)(x)$ for a.e. $x \in \Omega$ for some $f \in L^p(\Omega, \mathbb{R})$ if and only if $f \in L^{p/(p-1)}(\Omega, \mathbb{R})$ and $g = f - K\iota f$ (as equality in $L^{p/(p-1)}(\Omega, \mathbb{R})$). Since K is compact and ι is bounded, $K\iota$ is a compact operator from $L^{p/(p-1)}(\Omega, \mathbb{R})$ to itself. Since $K\iota$ is compact, we know that $\mathrm{id}_{L^{p/(p-1)}(\Omega,\mathbb{R})} - K\iota$ (where $\mathrm{id}_{L^{p/(p-1)}(\Omega,\mathbb{R})}$ shall denote here the identity operator on $L^{p/(p-1)}(\Omega, \mathbb{R})$) has closed range. Banach's closed range theorem (cf. Theorem 6.2.1 in M. Struwe's script) hence ensures that

$$\mathrm{im}(\mathrm{id}_{L^{p/(p-1)}(\Omega,\mathbb{R})} - K\iota) = {}^\perp \ker((\mathrm{id}_{L^{p/(p-1)}(\Omega,\mathbb{R})} - K\iota)^*).$$

Using the same notation as in (c), we know already that $K^* = J_p \tilde{K} J_{p/(p-1)}^{-1}$. For ι^*, we obtain for all $f, g \in L^{p/(p-1)}(\Omega, \mathbb{R})$ and $\varphi := J_p g \in (L^p(\Omega, \mathbb{R}))^*$ that

$$\begin{aligned}
(\iota^*\varphi)(f) = \varphi(\iota f) = (J_p g)(\iota f) &= \int_\Omega g(x)(\iota f)(x)\, dx \\
&= \int_\Omega g(x) f(x)\, dx = \int_\Omega (\iota g)(x) f(x)\, dx \\
&= (J_{p/(p-1)} \iota g)(f) = (J_{p/(p-1)} \iota J_p^{-1} \varphi)(f),
\end{aligned}$$

which leads to $\iota^* = J_{p/(p-1)} \iota J_p^{-1}$. This implies that

$$\begin{aligned}
(\mathrm{id}_{L^{p/(p-1)}(\Omega,\mathbb{R})} - K\iota)^* &= (\mathrm{id}_{L^{p/(p-1)}(\Omega,\mathbb{R})})^* - \iota^* K^* \\
&= \mathrm{id}_{(L^{p/(p-1)}(\Omega,\mathbb{R}))^*} - J_{p/(p-1)} \iota \tilde{K} J_{p/(p-1)}^{-1} \\
&= J_{p/(p-1)} (\mathrm{id}_{L^p(\Omega,\mathbb{R})} - \iota \tilde{K}) J_{p/(p-1)}^{-1}.
\end{aligned}$$

Hence, $\psi \in \ker((\mathrm{id}_{L^{p/(p-1)}(\Omega,\mathbb{R})} - K\iota)^*)$ if and only if $\psi = J_{p/(p-1)}\varphi$ for some $\varphi \in L^p(\Omega,\mathbb{R})$ and $\varphi \in \ker(\mathrm{id}_{L^p(\Omega,\mathbb{R})} - \iota\tilde{K})$. Above closed range theorem related considerations now imply that $g \in \mathrm{im}(\mathrm{id}_{L^{p/(p-1)}(\Omega,\mathbb{R})} - K\iota)$ if and only if it holds for all $\varphi \in \ker(\mathrm{id}_{L^p(\Omega,\mathbb{R})} - \iota\tilde{K})$ (which means nothing else than $\varphi \in L^p(\Omega)$ and $\varphi(x) = \int_\Omega k(y,x)\varphi(y)\,dy$ for a.e. $x \in \Omega$) that $0 = (J_{p/(p-1)}\varphi)(g) = \int_\Omega g\varphi\,dx$.

Exercise 4. 10(=2+4+4) points

Let $m \in \mathbb{N}$, let $p \in (1, \infty)$, let $s \in (0, \infty)$, let $\emptyset \neq \Omega \subseteq \mathbb{R}^m$ be a bounded open set, let $g \in L^{\frac{p}{p-1}}(\mathbb{R}^m, \mathbb{R})$, let $h \in L^p(\Omega, \mathbb{R})$, and let $V \colon L^p(\Omega, \mathbb{R}) \to \mathbb{R}$ and $E \colon L^p(\Omega, \mathbb{R}) \to \mathbb{R}$ be defined by

$$V(f) = \int_\Omega \int_\Omega g(x-y) f(y) f(x) \, dy \, dx \quad \text{for all } f \in L^p(\Omega, \mathbb{R})$$

and

$$E(f) = V(f) + \|f - h\|^s_{L^p(\Omega, \mathbb{R})} \quad \text{for all } f \in L^p(\Omega, \mathbb{R}).$$

(a) (2 points) Prove that V is well-defined.

Solution: For all $f \in L^p(\Omega, \mathbb{R})$, it holds by Hölder's inequality that

$$\begin{aligned}
\int_\Omega \int_\Omega |g(x-y) f(y) f(x)| \, dy \, dx &= \int_\Omega \int_\Omega |g(x-y) f(y)| \, dy \, |f(x)| \, dx \\
&\leq \int_\Omega \left[\int_\Omega |g(x-y)|^{\frac{p}{p-1}} \, dy \right]^{\frac{p-1}{p}} \|f\|_{L^p(\Omega, \mathbb{R})} |f(x)| \, dx \\
&\leq \|g\|_{L^{p/(p-1)}(\mathbb{R}^m, \mathbb{R})} \|f\|_{L^p(\Omega, \mathbb{R})} \int_\Omega |f(x)| \, dx \\
&\leq \|g\|_{L^{p/(p-1)}(\mathbb{R}^m, \mathbb{R})} \|f\|^2_{L^p(\Omega, \mathbb{R})} |\Omega|^{\frac{p-1}{p}} < \infty,
\end{aligned}$$

where $|\Omega|$ shall denote the Lebesgue measure of Ω. Thus, V is well-defined.

(b) (4 points) Prove that V is weakly sequentially continuous.

Solution: Let $(f_n)_{n \in \mathbb{N}} \subseteq L^p(\Omega, \mathbb{R})$ be a weakly converging sequence with weak limit $f_\infty \in L^p(\Omega, \mathbb{R})$. We have to show that $\limsup_{n \to \infty} |V(f_n) - V(f_\infty)| = 0$.

Due to $g \in L^{\frac{p}{p-1}}(\mathbb{R}^m, \mathbb{R})$ and the Hölder inequality, it holds for every $\varphi \in L^p(\Omega, \mathbb{R})$ and every $x \in \Omega$ that

$$\int_\Omega |g(x-y) \varphi(y)| \, dy \leq \|g\|_{L^{p/(p-1)}(\mathbb{R}^m, \mathbb{R})} \|\varphi\|_{L^p(\Omega, \mathbb{R})} < \infty.$$

Hence, the mapping $K \colon L^p(\Omega, \mathbb{R}) \to L^\infty(\Omega, \mathbb{R})$, given by

$$(K\varphi)(x) = \int_\Omega g(x-y) \varphi(y) \, dy \quad \text{for a.e. } x \in \Omega \text{ and all } \varphi \in L^p(\Omega, \mathbb{R}),$$

is well-defined, linear and bounded. Since Ω is bounded, it holds for every $\varphi \in L^p(\Omega, \mathbb{R})$ that $K\varphi \in L^{p/(p-1)}(\Omega, \mathbb{R})$. Moreover, since $f_n \xrightarrow{\mathrm{w}} f_\infty$ in $L^p(\Omega, \mathbb{R})$ as $n \to \infty$ and $g \in L^{p/(p-1)}(\Omega, \mathbb{R})$, we obtain for almost every $x \in \Omega$ that

$$\limsup_{n \to \infty} |(Kf_n)(x) - (Kf_\infty)(x)| = \limsup_{n \to \infty} \left| \int_\Omega g(x-y)(f_n(y) - f_\infty(y)) \, dy \right| = 0. \quad (5)$$

Furthermore, by the Banach–Steinhaus theorem and since $f_\infty \in \overline{\operatorname{conv}}(\{f_n \colon n \in \mathbb{N}\})$, we have $\|f_\infty\|_{L^p(\Omega,\mathbb{R})} \leq \sup_{n\in\mathbb{N}} \|f_n\|_{L^p(\Omega,\mathbb{R})} < \infty$. Therefore, we obtain for almost every $x \in \Omega$ that

$$\begin{aligned}\sup_{n\in\mathbb{N}} |(Kf_n)(x) - (Kf_\infty)(x)| &\leq \sup_{n\in\mathbb{N}} \left|\int_\Omega g(x-y)(f_n(y) - f_\infty(y))\,dy\right| \\ &\leq 2\|g\|_{L^{p/(p-1)}(\mathbb{R}^m,\mathbb{R})} \sup_{n\in\mathbb{N}} \|f_n\|_{L^p(\Omega,\mathbb{R})}.\end{aligned} \tag{6}$$

Since Ω is bounded, (equivalence classes of) constant functions belong to $L^{p/(p-1)}(\Omega,\mathbb{R})$. Properties (5) and (6) allow to apply Lebesgue's dominated convergence theorem to infer that

$$\limsup_{n\to\infty} \|Kf_n - Kf_\infty\|_{L^{p/(p-1)}(\Omega,\mathbb{R})} = 0. \tag{7}$$

Finally, we conclude by

$$\begin{aligned}\limsup_{n\to\infty} |V(f_n) - V(f_\infty)| &= \limsup_{n\to\infty} \left|\int_\Omega (Kf_n)(x) f_n(x)\,dx - \int_\Omega (Kf_\infty)(x) f_\infty(x)\,dx\right| \\ &\leq \limsup_{n\to\infty} \left|\int_\Omega ((Kf_n)(x) - (Kf_\infty)(x)) f_n(x)\,dx\right| \\ &\quad + \underbrace{\limsup_{n\to\infty} \left|\int_\Omega (Kf_\infty)(x)(f_n(x) - f_\infty(x))\,dx\right|}_{=0 \text{ since } Kf_\infty \in L^{p/(p-1)} \text{ and } f_n \xrightarrow{w} f_\infty \text{ in } L^p \text{ as } n\to\infty} \\ &\leq \underbrace{\limsup_{n\to\infty} \|Kf_n - Kf_\infty\|_{L^{p/(p-1)}(\Omega,\mathbb{R})}}_{=0 \text{ by } (7)} \underbrace{\sup_{k\in\mathbb{N}} \|f_k\|_{L^p(\Omega,\mathbb{R})}}_{<\infty \text{ by unif. bdd. princ.}} = 0.\end{aligned}$$

(c) (4 points) Prove that $E|_{\{f\in L^p(\Omega,\mathbb{R}) \colon f \geq 0 \text{ a.e.}\}}$ attains a global minimum under the additional assumption that $g \geq 0$ almost everywhere.

Solution: Since $g \geq 0$ a.e., it holds for every $f \in L^p(\Omega,\mathbb{R})$ with $f \geq 0$ a.e. that $V(f) \geq 0$, and, consequentially, $E(f) \geq 0$. Hence (keeping in mind that $\{f \in L^p(\Omega,\mathbb{R}) \colon f \geq 0 \text{ a.e.}\} \neq \emptyset$ as, e.g., 0 is contained), there exist $(f_n)_{n\in\mathbb{N}} \subseteq L^p(\Omega,\mathbb{R})$ satisfying $f_n \geq 0$ a.e. for all $n \in \mathbb{N}$ such that

$$\limsup_{n\to\infty} E(f_n) = \inf_{\varphi\in L^p(\Omega,\mathbb{R}), \varphi\geq 0 \text{ a.e.}} E(\varphi) \in [0,\infty).$$

Moreover, since it holds for every $\varphi \in L^p(\Omega,\mathbb{R})$ with $\varphi \geq 0$ a.e. that

$$E(\varphi) \geq \|\varphi - h\|^s_{L^p(\Omega,\mathbb{R})} \geq |\max\{\|\varphi\|_{L^p(\Omega,\mathbb{R})} - \|h\|_{L^p(\Omega,\mathbb{R})}, 0\}|^s,$$

we have that $E(\varphi) \to \infty$ as $\|\varphi\|_{L^p(\Omega,\mathbb{R})} \to \infty$ and, therefore, $\sup_{n\in\mathbb{N}} \|f_n\|_{L^p(\Omega,\mathbb{R})} < \infty$. Since $p \in (1,\infty)$, $L^p(\Omega,\mathbb{R})$ is reflexive and we may – passing to a subsequence if necessary – assume that $(f_n)_{n\in\mathbb{N}}$ converges weakly in $L^p(\Omega,\mathbb{R})$ to some (weak) limit f_∞. Since the set $\{\varphi \in L^p(\Omega,\mathbb{R})\colon \varphi \geq 0 \text{ a.e.}\}$ is (norm-)closed and convex, f_∞ also belongs to it (otherwise, the Hahn–Banach theorem could be used to construct a linear functional separating f_∞ from the closed convex hull of $\{f_n\colon n\in\mathbb{N}\}$, contradicting weak convergence). Finally, since V is weakly sequentially continuous by (b) and the norm is weakly sequentially lower semicontinuous, we conclude that

$$\begin{aligned}
\inf_{\varphi\in L^p(\Omega,\mathbb{R}),\varphi\geq 0 \text{ a.e.}} E(\varphi) &\leq E(f_\infty) \\
&= \underbrace{V(f_\infty)}_{=\lim_{n\to\infty} V(f_n) \text{ by (b)}} + \underbrace{\|f_\infty - h\|^s_{L^p(\Omega,\mathbb{R})}}_{\leq \liminf_{n\to\infty}\|f_n-h\|^s_{L^p(\Omega,\mathbb{R})}} \\
&\leq \liminf_{n\to\infty} \Big(V(f_n) + \|f_n - h\|^s_{L^p(\Omega,\mathbb{R})} \Big) \\
&= \liminf_{n\to\infty} E(f_n) = \inf_{\varphi\in L^p(\Omega,\mathbb{R}),\varphi\geq 0 \text{ a.e.}} E(\varphi),
\end{aligned}$$

which shows that f_∞ is a minimizer of $E|_{\{\varphi\in L^p(\Omega,\mathbb{R})\colon \varphi\geq 0 \text{ a.e.}\}}$.