1.1. Completeness, closedness, compactness, and metric spaces

Let (X, d) be a metric space. Prove the following statements:

(a) If $Y \subseteq X$ is a complete subspace (i.e., $Y \subseteq X$ and $(Y, d|_{Y\times Y})$ is complete), then Y is closed (i.e., a closed subset of X).

(b) If (X, d) is complete, then every closed subset $Y \subseteq X$ is complete (i.e., $(Y, d|_{Y\times Y})$ is complete).

(c) If (X, d) is compact, then (X, d) is complete.

1.2. Metrics on sequence spaces

Let (M, d) be a non-empty metric space. Consider the set of all M-valued sequences

$$S = \{(s_n)_{n\in\mathbb{N}} \mid \forall n \in \mathbb{N}\colon s_n \in M\}.$$

Let the function $\delta\colon S \times S \to [0, \infty)$ be defined by

$$\delta((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} 2^{-n} \frac{d(x_n, y_n)}{1 + d(x_n, y_n)}.$$

(a) Show that δ is a metric on S.

(b) Prove that (S, δ) is a complete metric space if (M, d) is a complete metric space.

1.3. Bounded metrics

Let (X, d) be a metric space and let $\mathcal{T}$ be the topology on X which is induced by d. Prove that there exists a bounded metric δ on X which induces the same topology $\mathcal{T}$.

1.4. Cantor's intersection theorem

The diameter of a subset A of a metric space (X, d) is defined by

$$\operatorname{diam}(A) = \sup(\{0\} \cup \{d(x, y) \mid x, y \in A\}).$$

(a) Prove that a metric space (X, d) is complete if and only if it holds for every nested sequence $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ of non-empty closed subsets $A_n \subseteq X$, $n \in \mathbb{N}$, with $\operatorname{diam}(A_n) \to 0$ for $n \to \infty$ that $\bigcap_{n\in\mathbb{N}} A_n \neq 0$. Moreover, prove that in this case $\bigcap_{n\in\mathbb{N}} A_n$ has exactly one element.

(b) Find an example of a complete metric space and a nested sequence of non-empty closed bounded subsets with empty intersection.

1.5. Intrinsic Characterisations

Let V be a vector space over $\mathbb{R}$. Prove the following equivalences.

(a) The norm $\|\cdot\|$ is induced by a scalar product $\langle\cdot,\cdot\rangle$ (in the sense that there exists a scalar product $\langle\cdot,\cdot\rangle$ such that $\forall x \in V: \ \|x\|^2 = \langle x, x\rangle$)

$\Leftrightarrow$ the norm satisfies the *parallelogram identity*, i. e. $\forall x, y \in V$:

$$\|x+y\|^2 + \|x-y\|^2 = 2\Big(\|x\|^2 + \|y\|^2\Big).$$

Hint. If $\|\cdot\|$ satisfies the parallelogram identity, consider $\langle x, y\rangle := \frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2$. Prove $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$ first for $\lambda \in \mathbb{N}$, then for $\lambda \in \mathbb{Q}$ and finally for $\lambda \in \mathbb{R}$.

(b) The metric $d(\cdot,\cdot)$ is induced by a norm $\|\cdot\|$ (in the sense that there exists a norm $\|\cdot\|$ such that $\forall x, y \in V: \ d(x,y) = \|x-y\|$)

$\Leftrightarrow$ the metric is *translation invariant* and *homogeneous*, i. e. $\forall v, x, y \in V \ \forall \lambda \in \mathbb{R}$:

$$d(x+v, y+v) = d(x,y),$$
$$d(\lambda x, \lambda y) = |\lambda| d(x,y).$$

1.6. A classic

Let (X, d) be a non-empty complete metric space, let $\lambda \in [0,1)$, and let $\Phi\colon X \to X$ be a mapping which satisfies for all $x, y \in X$ that $d(\Phi(x), \Phi(y)) \leq \lambda d(x,y)$. Show that there exists a unique $z \in X$ which satisfies $\Phi(z) = z$.