2.1. Statements of Baire

Definition. Let (M, d) be a metric space and $A \subseteq M$ a subset. Then, $\overline{A}$ denotes the closure, A° the interior and $A^\complement = M \setminus A$ the complement of A. We call A

- *dense*, if for every ball $B \subseteq M$, there is an element $x \in B \cap A$.
- *nowhere dense*, if $(\overline{A})^\circ = \emptyset$.
- *meagre*, if $A = \bigcup_{n\in\mathbb{N}} A_n$ is a countable union of nowhere dense sets A_n.
- *residual*, if $A^\complement$ is meagre.

Show that the following statements are equivalent.

(i) Every residual set $\Omega \subseteq M$ is dense in M.

(ii) The interior of every meagre set $A \subseteq M$ is empty.

(iii) The empty set is the only subset of M which is open and meagre.

(iv) Countable intersections of dense open sets are dense.

Hint. Show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i). Use that subsets of meagre sets are meagre and recall that $A \subseteq M$ is dense $\Leftrightarrow$ $\overline{A} = M$ $\Leftrightarrow$ $(M \setminus A)^\circ = \emptyset$.

Remark. Baire's theorem states that (i), (ii), (iii), (iv) are true if (M, d) is complete.

2.2. Algebraic (Hamel) bases for Banach spaces

Let X be a vector space. An *algebraic basis* for X is a subset $E \subseteq X$ such that every $x \in X$ is uniquely given as *finite* linear combination of elements in E.

(a) Show that, if $(X, \|\cdot\|)$ is a Banach space, then any algebraic basis for X is either finite or uncountable.

(b) Let $\mathcal{P}$ be the vector space of all real-valued polynomials over $\mathbb{R}$, i.e.,

$$\mathcal{P} = \left\{ p\colon \mathbb{R} \to \mathbb{R} \mid \exists n \in \mathbb{N}_0, a_0, a_1, \dots, a_n \in \mathbb{R}\colon \forall t \in \mathbb{R}\colon p(t) = \sum_{k=0}^{n} a_k t^k \right\}.$$

Show that there is no norm $\|\cdot\|\colon \mathcal{P} \to [0, \infty)$ on $\mathcal{P}$ turning $(\mathcal{P}, \|\cdot\|)$ into a Banach space.

2.3. An application to real analysis

Let $f \in C^0([0,\infty))$ be a continuous function satisfying

$$\forall t \in [0,\infty) : \lim_{n\to\infty} f(nt) = 0.$$

Prove that $\lim_{t\to\infty} f(t) = 0$.

2.4. Singularity condensation

Let $(X, \|\cdot\|_X)$ be a Banach space and let $(Y_1, \|\cdot\|_{Y_1}), (Y_2, \|\cdot\|_{Y_2}), \ldots$ be normed spaces. For every $n \in \mathbb{N}$, let $G_n \subseteq L(X, Y_n)$ be an unbounded set of linear continuous mappings from X to Y_n. Prove that there exists $x \in X$ satisfying for all $n \in \mathbb{N}$ that $\sup_{T\in G_n} \|Tx\|_{Y_n} = \infty$.

2.5. ℓ^p spaces

Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be a sequence. Define, for every $p \in [1,\infty]$,

$$\|(x_n)_{n\in\mathbb{N}}\|_{\ell^p} = \begin{cases} \left(\sum_{n\in\mathbb{N}} |x_n|^p\right)^{1/p} & \text{if } p < \infty, \\ \sup_{n\in\mathbb{N}} |x_n| & \text{if } p = \infty, \end{cases}$$

and let $\ell^p = \{(x_n)_{n\in\mathbb{N}} \mid \|(x_n)_{n\in\mathbb{N}}\|_{\ell^p} < \infty\}$.

(a) Show for every $p \in [1,\infty]$ that $(\ell^p, \|\cdot\|_{\ell^p})$ is a Banach space.

Let now $1 \le p < q \le \infty$. Prove that:

(b) $\ell^p \subsetneq \ell^q$ and $\|(x_n)_{n\in\mathbb{N}}\|_{\ell^q} \le \|(x_n)_{n\in\mathbb{N}}\|_{\ell^p}$ for every $(x_n)_{n\in\mathbb{N}} \in \ell^p$.

(c) ℓ^p is meager in ℓ^q.

(d) $\bigcup_{1\le r<q} \ell^r \subsetneq \ell^q$.

2.6. A reformulation of completeness for Banach spaces

Let $(X, \|\cdot\|)$ be a normed vector space. Prove that the following statements are equivalent.

(a) $(X, \|\cdot\|)$ is a Banach space.

(b) For every sequence $(x_n)_{n\in\mathbb{N}}$ in X with $\sum_{k=1}^{\infty} \|x_n\| < \infty$ the limit $\lim_{N\to\infty} \sum_{n=1}^{N} x_n$ exists.

2.7. Infinite-dimensional vector spaces and separability

(a) Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be an open set. Show that $L^p(\Omega)$ is an infinite-dimensional vector space for all $1 \leq p \leq \infty$.

(b) Let $(X, \mathcal{A}, \mu)$ be a measure space. Recall that if X is separable and the measure μ is finite (or, more generally, σ-finite) and if $1 \leq p < \infty$, then the space $L^p(X, \mathcal{A}, \mu)$ is separable. Roughly speaking, in the simple case when $X = (0,1)$, $\mathcal{A} =$ Borel-σ-algebra and $\mu = \mathcal{L}^1$, this relies on the fact that any element in those spaces can be arbitrarily well approximated by a function of the form

$$f = \sum_{i=1}^{k} q_i \, \chi_{B_i} \quad \text{for } k \in \mathbb{N},\ B_i := B_{r_i}(x_i),\ q_i \in \mathbb{Q},\ x_i \in \mathbb{Q} \cap (0,1),\ 0 < r_i \in \mathbb{Q}.$$

Show that instead $\left(L^\infty((0,1)), \|\cdot\|_{L^\infty((0,1))}\right)$ is *not* separable, i.e., it does not contain a countable dense subset.

(Recall that $\|u\|_{L^\infty((0,1))} := \inf\{K > 0 \mid |u(x)| \leq K \text{ for almost every } x \in (0,1)\}$.)