3.1. The space of bounded linear operators

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed $\mathbb{K}$-vector spaces with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $L(X, Y)$ be the space of bounded $\mathbb{K}$-linear operators $T: X \to Y$, equipped with the norm $\|\cdot\|_{L(X,Y)}: L(X, Y) \to [0, \infty)$, defined by

$$\|T\|_{L(X,Y)} = \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} \quad \text{for all } T \in L(X, Y).$$

(a) Prove that

$$\|T\|_{L(X,Y)} = \sup_{\|x\|_X \leq 1} \|Tx\|_Y = \sup_{\|x\|_X = 1} \|Tx\|_Y \quad \text{for all } T \in L(X, Y).$$

(b) Prove that $\|\cdot\|_{L(X,Y)}$ is indeed a norm on $L(X, Y)$.

(c) Prove that $(L(X, Y), \|\cdot\|_{L(X,Y)})$ is a $\mathbb{K}$-Banach space if and only if $(Y, \|\cdot\|_Y)$ is a $\mathbb{K}$-Banach space or $X = \{0\}$.

(d) Prove that the dual space $L(X, \mathbb{K})$ of X is complete.

3.2. Lipschitz functions

Let $X = \mathrm{Lip}([0, 1], \mathbb{R})$ be the $\mathbb{R}$-vector space of Lischitz continuous functions from $[0, 1]$ to $\mathbb{R}$ and let $Y = C^1([0, 1], \mathbb{R})$ be the $\mathbb{R}$-vector space of continuously differentiable functions from $[0, 1]$ to $\mathbb{R}$. Define the functions $\|\cdot\|_{\mathrm{Lip}}: X \to [0, \infty)$ and $\|\cdot\|_{\mathrm{C}^1}: Y \to [0, \infty)$ by

$$\|x\|_{\mathrm{Lip}} = \sup_{s \in [0,1]} |x(s)| + \sup_{\substack{s,t \in [0,1] \\ s \neq t}} \left| \frac{x(s) - x(t)}{s - t} \right| \quad \text{for all } x \in X,$$

$$\|y\|_{\mathrm{C}^1} = \sup_{s \in [0,1]} |x(s)| + \sup_{s \in [0,1]} |x'(s)| \quad \text{for all } y \in Y.$$

(a) Prove that $\|\cdot\|_{\mathrm{Lip}}$ is a norm on X.

(b) Show that $(X, \|\cdot\|_{\mathrm{Lip}})$ is an $\mathbb{R}$-Banach space.

(c) Demonstrate that $(Y, \|\cdot\|_{C^1})$ is isometrically embedded in $(X, \|\cdot\|_{\mathrm{Lip}})$ and that Y is closed in $(X, \|\cdot\|_{\mathrm{Lip}})$.

3.3. Completion of metric spaces

Let (X, d) be a metric space. A *completion* of (X, d) is a triple $(\mathbb{X}, \delta, \iota)$, where $(\mathbb{X}, \delta)$ is a complete metric space and $\iota: X \to \mathbb{X}$ is an isometric embedding with dense image.

(a) Let $(\mathbb{X}, \delta, \iota)$ be a completion of X. Then it satisfies the following universal property: whenever $\phi\colon X \to Y$ is 1-Lipschitz to a complete metric space (Y, d_Y) then there is a unique 1-Lipschitz map $\Phi\colon \mathbb{X} \to Y$ such that $\phi = \Phi \circ \iota$.

(b) If $(\mathbb{X}_1, \delta_1, \iota_1)$ and $(\mathbb{X}_2, \delta_2, \iota_2)$ are two completions of X, then there exists a unique isometric isomorphism $\psi\colon \mathbb{X}_1 \to \mathbb{X}_2$ such that $\iota_2 = \psi \circ \iota_1$.

(c) Prove the existence of a completion of (X, d). *Hint:* Recall that the space of continuous bounded real-valued functions $C_b(X, \mathbb{R})$ is a Banach space with respect to the norm $\|f\|_\infty = \sup_{x\in X} |f(x)|$. Fix $x_0 \in X$. For $y \in X$ let $f_y(x) = d(y, x) - d(x_0, x)$. Prove that $\iota(y) = f_y$ defines an isometric embedding $\iota\colon X \to C_b(X, \mathbb{R})$ and put $\mathbb{X} = \overline{\iota(X)}$.

3.4. Compactly supported sequences and their ℓ^∞-completion

Definition. We denote the space of compactly supported sequences by

$$c_c := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \exists N \in \mathbb{N}\ \forall n \geq N : \ x_n = 0\}$$

and the space of sequences converging to zero by

$$c_0 := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \lim_{n\to\infty} x_n = 0\}.$$

(a) Show that $(c_c, \|\cdot\|_{\ell^\infty})$ is *not* complete. What is a completion of this space?

(b) Prove the strict inclusion

$$\bigcup_{p=1}^{\infty} \ell^p \subsetneq c_0.$$

3.5. Operator norms need not be achieved

We consider the space $X = C^0([-1, 1], \mathbb{R})$ with its usual norm $\|\cdot\|_{C^0([-1,1])}$ and define

$$\begin{aligned} \varphi\colon X &\to \mathbb{R} \\ f &\mapsto \int_0^1 f(t)\,dt - \int_{-1}^0 f(t)\,dt. \end{aligned}$$

(a) Show that $\varphi \in L(X, \mathbb{R})$ with $\|\varphi\|_{L(X,\mathbb{R})} \leq 2$.

(b) Find a sequence $(f_n)_{n\in\mathbb{N}}$ in X such that $\|f_n\|_{C^0([-1,1])} = 1$ for every $n \in \mathbb{N}$ and such that $\varphi(f_n) \to 2$ as $n \to \infty$. This in fact implies $\|\varphi\|_{L(X,\mathbb{R})} = 2$.

(c) Prove that there does not exist $f \in X$ with $\|f\|_{C^0([-1,1])} = 1$ and $|\varphi(f)| = 2$.

3.6. Unbounded map and approximations

As in problem 3.4, we denote the space of compactly supported sequences by

$$c_c := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \exists N \in \mathbb{N}\ \forall n \geq N:\ x_n = 0\}$$

endowed with the norm $\|\cdot\|_{\ell^\infty}$. Consider the map

$$\begin{aligned} T\colon c_c &\to c_c \\ (x_n)_{n\in\mathbb{N}} &\mapsto (nx_n)_{n\in\mathbb{N}} \end{aligned}$$

(a) Show that T is not continuous.

(b) Construct continuous linear maps $T_m\colon c_c \to c_c$ such that

$$\forall x \in c_c: \quad T_m x \xrightarrow{m\to\infty} Tx.$$

3.7. Volterra equation

Let $k \in C([0,1]^2, \mathbb{R})$. The Volterra integral operator $T_k\colon C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is given by

$$(T_k f)(t) = \int_0^t k(t,s) f(s)\, ds \quad \text{for all } t \in [0,1], f \in C([0,1],\mathbb{R}).$$

(a) Prove that T_k is well-defined and continuous.

(b) For $\lambda \in \mathbb{R}$, let $\|\cdot\|_\lambda\colon C([0,1],\mathbb{R}) \to [0,\infty)$ be defined by $\|f\|_\lambda = \sup_{t\in[0,1]} e^{-\lambda t}|f(t)|$ for every $f \in C([0,1],\mathbb{R})$. Show that $\|\cdot\|_\lambda$ defines a norm equivalent to the supremum norm on $C([0,1],\mathbb{R})$.

(c) Estimate the operator norm of T_k on $(C([0,1],\mathbb{R}), \|\cdot\|_\lambda)$.

(d) Show that for every $g \in C([0,1],\mathbb{R})$ there exists a unique $f \in C([0,1],\mathbb{R})$ satisfying

$$\forall t \in [0,1]: \quad f(t) + \int_0^t k(t,s) f(s)\, ds = g(t).$$