4.1. Null and non-null limits

Denote by c the subspace of ℓ^∞ containing all the convergent sequences and let c_0 denote the subspace of sequences converging to 0.

(a) Prove that c is a closed subspace of ℓ^∞.

(b) Show that c is separable.

(c) Construct an isomorphism between c and c_0.

4.2. Baby Riesz representation

Let $p \in [1, \infty)$. Show that φ belongs to the dual space of ℓ^p, (i.e., $\varphi \in (\ell^p)' = L(\ell^p, \mathbb{R})$) if and only if there exists $(p_n)_{n\in\mathbb{N}} \in \ell^q$ such that

$$\varphi((x_n)_{n\in\mathbb{N}}) = \sum_{n=1}^{\infty} p_n x_n \quad \text{for all } (x_n)_{n\in\mathbb{N}} \in \ell^p,$$

where $q \in [1, \infty]$ is such that $\frac{1}{p} + \frac{1}{q} = 1$ (with the convention $\frac{1}{\infty} = 0$).

4.3. Infinite matrices

Consider the double sequence $(a_{jk})_{j,k\in\mathbb{N}}$ with $a_{jk} \in \mathbb{R}$ for every $j, k \in \mathbb{N}$.

(a) Let $\sup_{j,k\in\mathbb{N}}|a_{jk}| < \infty$ and let for every $x = (x_k)_{k\in\mathbb{N}} \in \ell^1$ the sequence Ax be given by

$$[Ax]_j = \sum_{k\in\mathbb{N}} a_{jk}x_k \quad \text{for all } j \in \mathbb{N}.$$

Show that this defines a bounded linear map from ℓ^1 to ℓ^∞ (i.e., $A \in L(\ell^1, \ell^\infty)$). Moreover, prove that $\|A\| = \sup_{j,k\in\mathbb{N}}|a_{jk}|$.

(b) Let $\sup_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}}|a_{jk}| < \infty$ and define for $x = (x_k)_{k\in\mathbb{N}} \in \ell^\infty$ the sequence Ax as above. Show that this defines a bounded linear map from ℓ^∞ to ℓ^∞ (i.e., $A \in L(\ell^\infty)$)). Moreover, prove that $\|A\| = \sup_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}}|a_{jk}|$.

4.4. The Fourier coefficients of functions in $L^1([0, 2\pi])$

For $f \in L^1([0, 2\pi])$, we define the k^{th} *Fourier coefficient* to be

$$\hat{f}(k) = \int_0^{2\pi} f(t)e^{-ikt}\,dt$$

and let $\mathcal{F}(f) = (\hat{f}(k))_{k\in\mathbb{Z}}$.

(a) Show that $\mathcal{F}\colon L^1([0,2\pi]) \to \ell^\infty(\mathbb{Z})$ defines a bounded linear operator.

(b) Prove the Riemann–Lebesgue lemma, that is, $\limsup_{|k|\to\infty}|\hat{f}(k)| = 0$ for all $f \in L^1([0,2\pi])$.

(c) Let $c_0(\mathbb{Z}) \subseteq \ell^\infty(\mathbb{Z})$ be the closed subspace of sequences converging to zero. Prove that $\mathcal{F}\colon L^1([0,2\pi]) \to c_0(\mathbb{Z})$ has dense range but is not onto.

4.5. Distance to closed subspaces

Let $(X, \|\cdot\|)$ be a normed $\mathbb{R}$-vector space and let $\varphi \in L(X,\mathbb{R})$ be an element of the dual space of X.

(a) Prove for every $x \in X$ that

$$\operatorname{dist}(x, \ker(\varphi)) = \frac{|\varphi(x)|}{\|\varphi\|_{L(X,\mathbb{R})}},$$

where $\operatorname{dist}(x, A) = \inf_{v\in A}\|x - v\|$ for $x \in X$ and $\emptyset \neq A \subseteq X$ denotes the distance of the point x to the set A.

Consider now the $\mathbb{R}$-vector space of continuous functions on the real half-line vanishing at ∞, i.e.,

$$C_0([0,\infty),\mathbb{R}) = \left\{ f \in C([0,\infty),\mathbb{R}) \mid \limsup_{t\to\infty} |f(t)| = 0 \right\},$$

equipped with the sup norm $\|\cdot\|_{\sup}$.

(b) Show that $H = \{f \in C_0([0,\infty),\mathbb{R}) \mid \int_0^\infty e^{-s} f(s)\,ds = 0\}$ is a closed subspace of the Banach space $(C([0,\infty),\mathbb{R}), \|\cdot\|_{\sup})$.

(c) Demonstrate that for every $f \in C_0([0,\infty),\mathbb{R}) \setminus H$, there is no $h \in H$ which realizes the distance, i.e., which satisfies $\operatorname{dist}(f, H) = \|f - h\|_{\sup}$.