5.1. Sums of closed subspaces

Show that the subspaces

$$U = \{(x_n)_{n\in\mathbb{N}} \in \ell^1 \mid \forall n \in \mathbb{N}: \ x_{2n} = 0\},$$
$$V = \{(x_n)_{n\in\mathbb{N}} \in \ell^1 \mid \forall n \in \mathbb{N}: \ x_{2n-1} = n x_{2n}\}$$

are both closed in $(\ell^1, \|\cdot\|_{\ell^1})$ while the subspace $U \oplus V$ is not closed in $(\ell^1, \|\cdot\|_{\ell^1})$.

Hint. For the second claim, show $c_c \subseteq U \oplus V$. (Recall c_c from problems 3.4 or 3.6.)

5.2. Vanishing boundary values

Let $X = C([0,1],\mathbb{R})$ and $U = C_0([0,1],\mathbb{R}) := \{f \in C([0,1],\mathbb{R}) \mid f(0) = 0 = f(1)\}$.

(a) Show that U is a closed subspace of X endowed with the norm $\|\cdot\|_X = \|\cdot\|_{C([0,1],\mathbb{R})}$.

(b) Compute the dimension of the quotient space X/U and find a basis for X/U.

5.3. Continuity of bilinear maps

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces. We consider the space $(X\times Y, \|\cdot\|_{X\times Y})$, where $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$ and a bilinear map $B\colon X\times Y \to Z$.

(a) Show that B is continuous if and only if

$$\exists C > 0 \quad \forall (x,y) \in X\times Y: \quad \|B(x,y)\|_Z \leq C\|x\|_X\|y\|_Y. \tag{†}$$

(b) Assume that $(X, \|\cdot\|_X)$ is complete. Assume further that the maps

$$\begin{aligned} X &\to Z \\ x &\mapsto B(x,y') \end{aligned} \qquad\qquad \begin{aligned} Y &\to Z \\ y &\mapsto B(x',y) \end{aligned}$$

are continuous for every $x' \in X$ and $y' \in Y$. Prove that then, (†) holds.

5.4. Diverging Fourier series

Prove for every $t \in [0, 2\pi]$ that there exists a continuous 2π-periodic function whose Fourier series does not converge at t.

5.5. Induced continuity

Let X and Y be Banach spaces, let Z be a metric space, let $T\colon X \to Y$ be linear, let $J\colon Y \to Z$ be injective and continuous, and let $J\circ T\colon X \to Z$ be continuous. Deduce that T is continuous.

5.6. Projections on closed convex sets

Let H be a Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$, let $C \subseteq H$ be a non-empty, closed and convex set, and let $x \in H$.

(a) Prove that there exists a unique $\xi \in C$ satisfying $\|x-\xi\| = \inf_{y\in C}\|x-y\|$.

(b) Prove for all $y \in C$ the following equivalence:

$$\Big(\|x-y\| = \inf_{z\in C}\|x-z\|\Big) \Leftrightarrow \Big(\operatorname{Re}\langle x-y, z-y\rangle \leq 0 \quad \text{for all } z \in C\Big).$$

5.7. Hardy space

Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ and set

$$\mathcal{H}^2(\mathbb{D}) = \left\{ f \colon \mathbb{D} \to \mathbb{C} \mid f \text{ holomorphic with } \sup_{0\leq r<1} \int_0^{2\pi} |f(re^{it})|^2\,dt < \infty \right\}.$$

(a) Derive a characterization of all the functions $f \in \mathcal{H}^2(\mathbb{D})$ in terms of the coefficients $(a_k(f))_{k\in\mathbb{N}_0} \subseteq \mathbb{C}$ of its power series expansion.

(b) Demonstrate that, for all $f, g \in \mathcal{H}^2(\mathbb{D})$, the limit

$$\langle f, g\rangle := \lim_{r\to 1} \int_0^{2\pi} f(re^{it})\overline{g(e^{it})}\,\frac{dt}{2\pi}$$

exists and express it in terms of the coefficients $(a_k(f))_{k\in\mathbb{N}_0}, (a_k(g))_{k\in\mathbb{N}_0} \subseteq \mathbb{C}$ of their power series expansions.

(c) Prove that $(\mathcal{H}^2(\mathbb{D}), \langle\cdot,\cdot\rangle)$ is a Hilbert space with $(\mathbb{D} \ni z \mapsto z^n \in \mathbb{C})_{n\in\mathbb{N}_0} \subseteq \mathcal{H}^2(\mathbb{D})$ being an orthonormal basis.