6.1. Topological complement

Definition. Let $(X, \|\cdot\|_X)$ be a Banach space. A subspace $U \subseteq X$ is called *topologically complemented* if there is a subspace $V \subseteq X$ such that the linear map I given by

$$I\colon (U \times V, \|\cdot\|_{U\times V}) \to (X, \|\cdot\|_X), \qquad \|(u,v)\|_{U\times V} := \|u\|_X + \|v\|_X,$$
$$(u,v) \mapsto u + v$$

is a continuous isomorphism of normed spaces with continuous inverse. In this case V is said to be a *topological complement* of U.

(a) Prove that $U \subseteq X$ is topologically complemented if and only if there exists a continuous linear map $P\colon X \to X$ with $P \circ P = P$ and image $P(X) = U$.

(b) Show that a topologically complemented subspace must be closed.

6.2. Heavily diverging Fourier series

Let $X = \{f \in C([0, 2\pi], \mathbb{R})\colon f(0) = f(2\pi)\}$. For $m \in \mathbb{N}_0$ and $f \in X$ we denote the m^{th} partial sum of the Fourier series by $S_m f$, that is,

$$(S_m f)(t) = \sum_{k=-m}^{m} \left[\frac{1}{2\pi}\int_0^{2\pi} f(s)e^{-iks}\, ds\right] e^{ikt}.$$

This exercise's goal is to prove the existence of a continuous 2π-periodic function whose Fourier series does not converge at uncountably many points. To this end, let $\{t_k\colon k \in \mathbb{N}\} \subseteq [0, 2\pi]$ be dense.

(a) Prove that there exists $f_0 \in X$ such that $\sup_{m\in\mathbb{N}} |(S_m f_0)(t_n)| = \infty$ for all $n \in \mathbb{N}$.

(b) Show for every $k \in \mathbb{N}$ that $\{t \in [0, 2\pi]\colon |(S_m f_0)(t)| \le k \text{ for all } m \in \mathbb{N}_0\}$ is closed and meagre.

(c) Conclude that there is an uncountable subset of $[0, 2\pi]$ on which the Fourier series of f_0 does not converge.

6.3. The Fundamental Principles Fail for Non-Complete Spaces

Consider the vector space c_c of real sequences $x = (x_n)_{n\in\mathbb{N}}$ with only finitely many non-zero terms (cf. problems 3.4 and 3.6 as well as problem 5.1). Let $\|x\|_{\ell^1} = \sum_{n=1}^{\infty} |x_n|$ and $\|x\|_{\ell^\infty} = \sup_{n\in\mathbb{N}} |x_n|$ be the ℓ^1 and ℓ^∞ norms, respectively.

(a) The family of linear functionals $\varphi_m\colon c_c \to \mathbb{R}$ given by $\varphi_m(x) = m x_m$, $m \in \mathbb{N}$, is pointwise bounded, but not uniformly bounded (in either norm on c_c).

(b) The identity operator $(c_c, \|\cdot\|_{\ell^1}) \to (c_c, \|\cdot\|_{\ell^\infty})$ is continuous, but not open.

(c) The identity operator $(c_c, \|\cdot\|_{\ell^\infty}) \to (c_c, \|\cdot\|_{\ell^1})$ has closed graph, but is not continuous.

6.4. Zabreiko's Lemma

Let $(X, \|\cdot\|)$ be a $\mathbb{K}$-Banach space (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$), let $p\colon X \to [0,\infty)$ be a *seminorm* (that is, for all $x, y \in X$, $\lambda \in \mathbb{K}$ it holds that $p(x+y) \leq p(x) + p(y)$ and $p(\lambda x) = |\lambda| p(x)$), and assume that

$$p\left(\sum_{k=1}^{\infty} x_k\right) \leq \sum_{k=1}^{\infty} p(x_k) \quad \text{for all } (x_k)_{k\in\mathbb{N}} \subseteq X \text{ for which } \sum_{k=1}^{\infty} x_k \text{ converges.}$$

Demonstrate that there exists $M \in [0,\infty)$ such that

$$p(x) \leq M\|x\| \quad \text{for all } x \in X.$$

This is *Zabreiko's lemma. Hint:* Mimick the proof of the open mapping theorem.

6.5. Proving everything by Zabreiko's lemma

Recall Zabreiko's lemma from problem 6.4. In this problem we will infer more or less all the fundamental principles from Zabreiko's lemma. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

(a) (*Uniform boundedness principle.*) For a $\mathbb{K}$-Banach space $(X, \|\cdot\|_X)$, a normed $\mathbb{K}$-vector space $(Y, \|\cdot\|_Y)$ and a collection of continuous linear mappings $\mathcal{F} \subseteq L(X,Y)$, prove (by applying Zabreiko's lemma) that

$$\left(\sup_{T\in\mathcal{F}} \|Tx\|_Y < \infty \quad \text{for every } x \in X\right) \Rightarrow \sup_{T\in\mathcal{F}} \|T\|_{L(X,Y)} < \infty.$$

(b) (*Closed graph theorem.*) For $\mathbb{K}$-Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ and a linear map $T\colon X \to Y$, prove (by applying Zabreiko's lemma) that

$$\left(\operatorname{graph}(T) = \{(x, Tx) \mid x \in X\} \subseteq X \times Y \text{ is closed}\right) \Rightarrow T \in L(X,Y).$$

(c) (*Open mapping theorem.*) For $\mathbb{K}$-Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ and a surjective continuous linear map $T \in L(X,Y)$, prove (by applying Zabreiko's lemma) that T is open.

6.6. Riesz representation theorem for Hilbert spaces

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let $(H, \langle\cdot,\cdot\rangle)$ be $\mathbb{K}$-Hilbert space.

(a) Prove for every $\varphi \in L(H, \mathbb{K})$ (i.e., every φ in the dual space of H) that there exists a unique $v \in H$ such that

$$\varphi(u) = \langle u, v\rangle \quad \text{for every } u \in H.$$

(b) Prove that the map $T\colon H \to L(H, \mathbb{K})$, defined by

$$(Tv)(u) = \langle u, v\rangle \quad \text{for all } u, v \in H,$$

is antilinear, bijective and isometric.

6.7. Reproducing kernels

Let S be a set and let H be a $\mathbb{K}$-Hilbert space (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) of functions on S. A *reproducing kernel* for H is a function $k\colon S \times S \to \mathbb{K}$ satisfying for all $t \in S$, $f \in H$ that $k_t = (S \ni s \mapsto k(s,t) \in \mathbb{K}) \in H$ and $f(t) = \langle f, k_t\rangle$.

(a) Prove that a reproducing kernel, if existent, is unique.

(b) Show that a reproducing kernel exists if and only if, for every $t \in S$, the mapping $H \ni f \mapsto f(t) \in \mathbb{K}$ is continuous.

(c) Prove that $H = \overline{\operatorname{span}\{k_t \mid t \in S\}}$ if a reproducing kernel exists.

(d) Prove that the Hardy space $\mathcal{H}^2(\mathbb{D})$ (cf. problem 5.7) possesses a reproducing kernel and determine the reproducing kernel for $\mathcal{H}^2(\mathbb{D})$.