7.1. Finite-dimensional subspaces are topologically complemented

Let $(X, \|\cdot\|_X)$ be a $\mathbb{K}$-Banach space (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) and $U \subseteq X$ a closed subspace. Show that:

(a) If $\dim(U) < \infty$, then U is topologically complemented.

(b) If $\dim(X/U) < \infty$, then U is topologically complemented.

7.2. Dual spaces of c_0 and c

Recall the ($\mathbb{R}$-vector) spaces

$$c_0 := \Big\{(x_k)_{k\in\mathbb{N}} \in \ell^\infty \;\Big|\; \lim_{k\to\infty} x_k = 0\Big\}, \qquad c := \Big\{(x_k)_{k\in\mathbb{N}} \in \ell^\infty \;\Big|\; \lim_{k\to\infty} x_k \text{ exists}\Big\}.$$

with norm $\|\cdot\|_{\ell^\infty}$ (cf. problems 3.4 and 4.1).

(a) Show that the dual space of $(c_0, \|\cdot\|_{\ell^\infty})$ is *isometrically* isomorphic to $(\ell^1, \|\cdot\|_{\ell^1})$.

(b) To which space is the dual space of $(c, \|\cdot\|_{\ell^\infty})$ isomorphic?

7.3. Banach Limits

Define the shift operator T on (the $\mathbb{R}$-Banach space) $\ell^\infty = \ell^\infty(\mathbb{N}, \mathbb{R})$ by

$$Ty = (y_{n+1})_{n\in\mathbb{N}} \quad \text{for all } y = (y_n)_{n\in\mathbb{N}} \in \ell^\infty.$$

Consider the subspace $X = \{x \in \ell^\infty \mid \exists y \in \ell^\infty \text{ s.t. } x = y - Ty\}$.

(a) The closure of X contains the space of sequences that converge to zero.

(b) Let c be the constant sequence $c = (1)_{n\in\mathbb{N}}$. Show that $\operatorname{dist}(c, X) = 1$ where $\operatorname{dist}(c, X) = \inf_{x\in X} \|c - x\|_{\ell^\infty}$.

(c) By the Hahn–Banach theorem there is a linear functional $L\colon \ell^\infty \to \mathbb{R}$ such that $L(c) = 1$, $\|L\|_{L(X,\mathbb{R})} = 1$ and $L(x) = 0$ for all $x \in X$.

(i) Show that $L(Ty) = L(y)$ for all $y \in \ell^\infty$.

(ii) Verify that $L(y) \geq 0$ whenever $y \geq 0$ (in the sense that, for $y = (y_n)_{n\in\mathbb{N}} \in \ell^\infty$, it holds that $y_n \geq 0$ for all $n \in \mathbb{N}$) and deduce that $\liminf_{n\to\infty} y_n \leq L(y) \leq \limsup_{n\to\infty} y_n$ for all $y \in \ell^\infty$. It follows that $L(y) = \lim_{n\to\infty} y_n$ whenever y is convergent.

(iii) Find y and z such that $L(yz) \neq L(y)L(z)$.

(iv) Show that there is no $z \in \ell^1$ such that $L(y) = \sum_{n=1}^{\infty} y_n z_n$ for all $y = (y_n)_{n\in\mathbb{N}} \in \ell^\infty$, so L is a functional in $(\ell^\infty)^* \setminus \ell^1$.

7.4. Inseparable Disjoint Closed Convex Sets

In the Hilbert space $\ell^2 = \ell^2(\mathbb{N}, \mathbb{R})$ of square summable sequences, set $A = \mathbb{R}e_1$ and let

$$B = \left\{ x \in \ell^2 \colon x_1 \geq n \cdot \left| x_n - \frac{1}{n^{2/3}} \right| \text{ for all } n \geq 2 \right\}.$$

(a) Verify that A and B are disjoint, non-empty, closed and convex.

(b) Prove that $A - B$ is dense in ℓ^2 and conclude that there is no non-zero continuous linear functional on ℓ^2 which separates A from B.

7.5. Strict convexity and uniqueness of the Hahn–Banach extension

(a) (*Ruston's Theorem*) Show that the following properties of a normed $\mathbb{R}$-vector space $(X, \|\cdot\|_X)$ are equivalent:

(i) If $x \neq y$ and $\|x\|_X = 1 = \|y\|_X$ then $\|\frac{x+y}{2}\|_X < 1$.

(ii) If $x \neq 0 \neq y$ and $\|x + y\|_X = \|x\|_X + \|y\|_X$, then $x = \lambda y$ for some $\lambda > 0$.

(iii) If $\varphi \in X^*$ is a nonzero bounded linear functional, then there is at most one $x \in X$ with $\|x\|_X = 1$ such that $\varphi(x) = \|\varphi\|_{X^*}$.

Remark: A normed space is said to be *strictly convex* if any of these properties is satisfied. Point (i) says that the unit sphere contains no non-trivial line segment. Point (ii) says that equality in the triangle inequality only occurs in the trivial situation. Point (iii) says that for $\varphi \neq 0$ the *support hyperplane* $H_\varphi = \{x \in X \colon \varphi(x) = \|\varphi\|_{X^*}\}$ of the unit sphere meets the sphere in at most one point (note that $\inf_{x \in H_\varphi} \|x\|_X = 1$).

(b) For which $p \in [1, \infty]$ is $L^p([0, 1], \mathbb{R})$ strictly convex?

(c) Is $C([0, 1], \mathbb{R})$ strictly convex?

(d) If X^* is strictly convex, then every bounded linear functional ψ defined on a subspace U of X has a unique extension Ψ to all of X such that $\|\Psi\|_{X^*} = \|\psi\|_{L(U,\mathbb{R})}$.

7.6. Another application of the Hahn–Banach theorem

Let $(X, \|\cdot\|_X)$ be a normed $\mathbb{K}$-vector space (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$), let $(x_j)_{j \in \mathbb{N}} \subseteq X$ be a sequence of points in X, let $\gamma \in [0, \infty)$ and let $(\alpha_j)_{j \in \mathbb{N}} \subseteq \mathbb{K}$ be a sequence. Prove that the following are equivalent:

(i) There exists a functional $l \in X^*$ satisfying

$$\|l\|_{X^*} \leq \gamma \quad \text{and} \quad l(x_j) = \alpha_j \quad \text{for all } j \in \mathbb{N}.$$

(ii) It holds that

$$\left|\sum_{j=1}^{n} \beta_j \alpha_j\right| \leq \gamma \left\|\sum_{j=1}^{n} \beta_j x_j\right\|_X \quad \text{for all } n \in \mathbb{N}, (\beta_j)_{j=1}^{n} \subseteq \mathbb{K}.$$