### 8.1. Strict convexity and extremal points

A normed space $(X, \|\cdot\|)$ with $X \neq \{0\}$ is strictly convex (cf. Problem 7.5) if and only if the unit sphere $S := \{x \in X : \|x\| = 1\}$ is equal to the set of extremal points of the closed unit ball $B := \{x \in X : \|x\| \leq 1\}$.

### 8.2. Closedness/Non-closedness of sets of extremal points

**(a)** Let $K \subseteq \mathbb{R}^2$ be a closed convex subset. Prove that the set $E$ of all extremal points of $K$ is closed.

**(b)** Consider the convex hull $C$ of the circle $\{(1 + \cos(\varphi), \sin(\varphi), 0) : 0 \leq \varphi \leq 2\pi\}$ and the points $(0, 0, \pm 1)$ in $\mathbb{R}^3$. Determine the extremal points of C.

### 8.3. Birkhoff–von Neumann theorem

A matrix $M = (M_{ij})_{1\leq i,j\leq n} \in \mathbb{R}^{n\times n}$ (where $n \in \mathbb{N}$) with $M_{ij} \geq 0$ is called *doubly stochastic* iff its rows and columns all add up to one: $\sum_{i=1}^{n} M_{ij} = 1 = \sum_{i=1}^{n} M_{ji}$. Prove that every doubly stochastic matrix is a convex combination of permutation matrices.

*Hint*: Suppose $M$ is a doubly stochastic matrix. Find a permutation matrix $P$ and $\lambda \in (0, \infty)$ such that $N = M - \lambda P$ has non-negative entries. If $N \geq 0$ then $\frac{1}{1-\lambda} N$ is doubly stochastic. One way to find $P$ is as follows:

Recall **Hall's Marriage Theorem**: Assume $X$ and $Y$ are finite sets and let $\Gamma \subseteq X \times Y$. The following statements are equivalent:

(i) There exists an injective function $f : X \to Y$ whose graph is contained in $\Gamma$.

(ii) For every $A \subseteq X$ the set $\Gamma(A) = \{y \in Y \mid (x, y) \in \Gamma \text{ for some } x \in A\}$ satisfies $\#\Gamma(A) \geq \#A$.

Let $X = Y = \{1, 2, \ldots, n\}$ and let $\Gamma = \{(i, j) \in X \times Y \mid M_{ij} > 0\}$. Use the fact that $M$ is doubly stochastic to verify condition (ii). The injective map $f : X \to Y$ from (i) determines a permutation of $\{1, 2, \ldots, n\}$.

### 8.4. Topologies induced by linear functionals

Let $X$ be a real vector space.

**(a)** Let $n \in \mathbb{N}$ and let $\varphi_1, \varphi_2, \ldots, \varphi_n, \psi : X \to \mathbb{R}$ be linear functionals. Prove that the following are equivalent:

(i) There exist $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ satisfying $\psi = \sum_{k=1}^{n} \lambda_k \varphi_k$.

(ii) There is a constant $C \in (0,\infty)$ such that $|\psi(x)| \leq C \max_{1\leq k\leq n}|\varphi_k(x)|$ for all $x \in X$.

(iii) $\ker(\psi) \supseteq \bigcap_{k=1}^{n} \ker(\varphi_k)$.

**(b)** Let $F \subseteq \{f\colon X \to \mathbb{R} \mid f \text{ is linear}\}$ be a family of linear functionals and let $\mathcal{U}_F$ be the topology on $X$ induced by $F$. Prove that

$$\operatorname{span}(F) = \{\varphi\colon X \to \mathbb{R} \mid \varphi \text{ is } \mathcal{U}_F\text{-continuous and linear}\}.$$

**(c)** Suppose $X$ is a normed space. Consider a weak*-continuous linear functional $\varphi\colon X^* \to \mathbb{R}$. Prove that there is $x \in X$ such that $\varphi(f) = f(x)$ for all $f \in X^*$.

### 8.5. Weak topologies

Let $(X, \|\cdot\|_X)$ be a normed space and let $\tau_{\mathrm{w}}$ denote the weak topology on $X$. This exercise's goal is to show that $\tau_{\mathrm{w}}$ is not metrizable if $X$ is infinite-dimensional. Let us start by recalling what a *neighbourhood basis* is and what it means for a topology to be *metrizable*:

- (*Neighbourhood basis*) Let $(Y,\tau)$ be a topological space. Denoting the set of all neighbourhoods of a point $y \in Y$ by

  $$\mathcal{U}_y = \{U \subseteq Y \mid \exists\, O \in \tau:\ y \in O \subseteq U\},$$

  we call $\mathcal{B}_y \subseteq \mathcal{U}_y$ a *neighbourhood basis* of $y$ in $(Y,\tau)$, if $\forall\, U \in \mathcal{U}_y\ \exists V \in \mathcal{B}_y\colon V \subseteq U$.

- (*Metrizability*) A topological space $(Y,\tau)$ is called *metrizable* if there exists a metric $d\colon Y \times Y \to \mathbb{R}$ on $Y$ such that, denoting $B_\varepsilon(a) = \{y \in Y \mid d(y,a) < \varepsilon\}$ (for $a \in Y$, $\varepsilon \in (0,\infty)$), there holds

  $$\tau = \{O \subseteq Y \mid \forall a \in O\ \exists\, \varepsilon > 0:\ B_\varepsilon(a) \subseteq O)\}.$$

**(a)** Show that any metrizable topology $\tau$ satisfies the *first axiom of countability* which means that each point has a *countable* neighbourhood basis.

**(b)** Prove that

$$\mathcal{B} := \left\{ \bigcap_{k=1}^{n} f_k^{-1}\Big((-\varepsilon,\varepsilon)\Big) \,\middle|\, n \in \mathbb{N},\ f_1,\dots,f_n \in X^*,\ \varepsilon > 0 \right\}$$

is a neighbourhood basis of $0 \in X$ in $(X,\tau_{\mathrm{w}})$.

**(c)** Show that if $(X, \tau_w)$ is first countable, then $(X^*, \|\cdot\|_{X^*})$ admits a countable algebraic basis.

*Hint:* Problems 8.4(a) and (b).

**(d)** Assume that $X$ is infinite-dimensional and conclude from (a), (c) and Problem 2.2 (*Algebraic bases for Banach spaces*) that $(X, \tau_w)$ is not metrizable.

### 8.6. Weak and weak* topology on $\ell^1$

Let $e_n = (\delta_{kn})_{k\in\mathbb{N}} \subseteq \mathbb{R}$ for every $n \in \mathbb{N}$. For $p \in (1, \infty)$, $(e_n)_{n\in\mathbb{N}} \subseteq \ell^p$ converges to 0 with respect to both weak as well as weak* convergence in $\ell^p$ as $n \to \infty$. $\ell^1$ behaves similarly with respect to weak* convergence, but differently with respect to weak convergence:

**(a)** Show that $(e_n)_{n\in\mathbb{N}} \subseteq \ell^1$ does not converge weakly to 0 in $\ell^1$.

**(b)** Viewing $\ell^1$ as the dual space of $c_0$ (cf. Problem 7.2 (*Dual spaces of* $c_0$ *and* $c$)), argue that $(e_n)_{n\in\mathbb{N}}$ converges to zero in the weak* topology.

**(c)** (*Schur's Theorem.*) Let $(x_n)_{n\in\mathbb{N}} \subseteq \ell^1$ be converging weakly to 0. Prove that $\|x_n\|_{\ell^1} \to 0$ as $k \to \infty$.