9.1. Metrizability and weak* topology

Let $(X, \|\cdot\|_X)$ be a separable normed $\mathbb{K}$-vector space (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$). Prove that the weak* topology on the unit ball $B^* := \{\varphi \in X^* \colon \|\varphi\|_{X^*} \leq 1\}$ of X^* is metrizable.

9.2. Weak convergence in Hilbert spaces

Let $(H, (\cdot,\cdot)_H)$ be an infinite-dimensional $\mathbb{K}$-Hilbert space (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$).

(a) Let $(x_n)_{n\in\mathbb{N}} \subseteq H$ and $x_\infty \in H$ satisfy that $x_n \xrightarrow{w} x_\infty$ in H and $\|x_n\|_H \to \|x_\infty\|_H$ in $\mathbb{R}$ as $n \to \infty$. Prove that $x_n \to x_\infty$ in H as $n \to \infty$, i. e. $\limsup_{n\to\infty} \|x_n - x_\infty\|_H = 0$.

(b) Suppose $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subseteq H$ and $x_\infty, y_\infty \in H$ satisfy that $x_n \xrightarrow{w} x_\infty$ and $\|y_n - y_\infty\|_H \to 0$ as $n \to \infty$. Prove that $(x_n, y_n)_H \to (x_\infty, y_\infty)_H$ as $n \to \infty$.

(c) Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal system of $(H, (\cdot,\cdot)_H)$. Prove $e_n \xrightarrow{w} 0$ as $n \to \infty$.

(d) Given any $x_\infty \in H$ with $\|x_\infty\|_H \leq 1$, prove that there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in H satisfying $\|x_n\|_H = 1$ for all $n \in \mathbb{N}$ and $x_n \xrightarrow{w} x_\infty$ as $n \to \infty$.

(e) Let the functions $f_n \colon [0, 2\pi] \to \mathbb{R}$ be given by $f_n(t) = \sin(nt)$ for $n \in \mathbb{N}$. Prove the Riemann–Lebesgue Lemma: $f_n \xrightarrow{w} 0$ in $L^2([0, 2\pi], \mathbb{R})$ as $n \to \infty$.

9.3. Annihilating annihilators

Let X be a normed $\mathbb{K}$-vector space (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$).

- For every set $U \subseteq X$ let $U^\perp \subseteq X^*$ be defined by $U^\perp = \{\varphi \in X^* \colon \varphi(u) = 0 \text{ for all } u \in U\}$.
- For every set $\Phi \subseteq X^*$ let ${}^\perp\Phi \subseteq X$ be defined by ${}^\perp\Phi = \{x \in X \colon \varphi(x) = 0 \text{ for all } \varphi \in \Phi\}$.

Prove for all $\emptyset \neq U \subseteq X$ and $\emptyset \neq \Phi \subseteq X^*$ that ${}^\perp(U^\perp) = \overline{\operatorname{span}(U)}$ and $\overline{\operatorname{span}(\Phi)} \subseteq ({}^\perp\Phi)^\perp$.

9.4. Duals and quotient spaces

Let $(X, \|\cdot\|_X)$ be a normed $\mathbb{K}$-vector space (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) and $U \subseteq X$ a closed subspace.

(a) Prove that $(X/U)^*$ is isometrically isomorphic to $U^\perp$.

(b) Prove that U^* is isometrically isomorphic to $X^*/U^\perp$.

(c) Prove that reflexivity of X implies reflexivity of U (in other words, closed subspaces of reflexive spaces are reflexive).

9.5. Invariant measures à la Krylov–Bogolioubov

Let (K, d) be a non-empty compact metric space and let $T\colon K \to K$ be continuous. Prove that there exists a Borel probability measure $\mu \in \mathcal{P}(K)$ on K satisfying for all Borel sets $A \subseteq K$ that $\mu(T^{-1}(A)) = \mu(A)$.

Hint: Use Problem 7.3 (*Banach limits*) to show that there exists $\varphi \in (C(K,\mathbb{R}))^*$ satisfying $\varphi \geq 0$, $\|\varphi\|_{(C(K,\mathbb{R}))^*} = 1$ and $\varphi(f) = \varphi(f \circ T)$ for all $f \in C(K, \mathbb{R})$. Conclude recalling **Riesz's representation theorem**:

With (K, d) being a compact metric space and with $\mathcal{M}(K)$ denoting the set of Borel regular finite signed measures on K, $\mathcal{M}(K)$ is isometrically isomorphic to $(C(K,\mathbb{R}))^*$ via the mapping $\Phi\colon \mathcal{M}(K) \to (C(K,\mathbb{R}))^*$, defined by

$$[\Phi(\mu)](f) = \int_K f\,d\mu \quad \text{for all } \mu \in \mathcal{M}(K), f \in C(K,\mathbb{R}).$$

In particular, the positive regular Borel measures correspond to the positive continuous linear functionals.

9.6. Optimal transport à la Kantorovich

Let (X, d_X) and (Y, d_Y) be compact metric spaces, let $c\colon X \times Y \to \mathbb{R} \cup \{\infty\}$ be lower semi-continuous, and let $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$ be probability measures on X and Y, respectively. We denote by $\Gamma(\mu, \nu)$ the set of probability measures on $X \times Y$ with first marginal μ and second marginal ν, i.e.,

$$\Gamma(\mu,\nu) = \left\{ \gamma \in \mathcal{P}(X \times Y)\colon \begin{array}{c} \gamma(A \times Y) = \mu(A), \gamma(X \times B) = \nu(B) \\ \text{for all Borel sets } A \subseteq X, B \subseteq Y \end{array} \right\}.$$

Prove that there exists $\gamma \in \Gamma(\mu, \nu)$ satisfying that

$$\int_{X\times Y} c(x,y)\,d\gamma(x,y) = \inf_{\eta \in \Gamma(\mu,\nu)} \int_{X \times Y} c(x,y)\,d\eta(x,y).$$

Hint: Assume first that c is continuous. For general lower semi-continuous c, use that c can be written as pointwise limit of an increasing sequence $(f_k)_{k\in\mathbb{N}} \subseteq C(X \times Y, \mathbb{R})$.

9.7. Minimal Energy

Let $m \in \mathbb{N}$ and let $\Omega \subseteq \mathbb{R}^m$ be a bounded measurable set with $|\Omega| > 0$. For $g \in L^2(\mathbb{R}^m)$, we define the map

$$\begin{aligned} V\colon L^2(\Omega) &\to \mathbb{R} \\ f &\mapsto \int_\Omega \int_\Omega g(x-y) f(x) f(y)\, dy\, dx \end{aligned}$$

and given $h \in L^2(\Omega)$, we define the map

$$\begin{aligned} E\colon L^2(\Omega) &\to \mathbb{R} \\ f &\mapsto \|f-h\|^2_{L^2(\Omega)} + V(f). \end{aligned}$$

(a) Prove that V is weakly sequentially continuous.

(b) Under the assumption $g \geq 0$ almost everywhere, prove that E restricted to

$$L^2_+(\Omega) := \{f \in L^2(\Omega) \mid f(x) \geq 0 \text{ for almost every } x \in \Omega\}$$

attains a global minimum.

9.8. Lions–Stampacchia

Let $(H, (\cdot,\cdot)_H)$ be a real Hilbert space and let $a\colon H \times H \to \mathbb{R}$ be a bilinear map so that:

(i) $a(x,y) = a(y,x)$ for every $x, y \in H$,

(ii) there exists $\Lambda \in (0,\infty)$ so that $|a(x,y)| \leq \Lambda \|x\|_H \|y\|_H$ for every $x, y \in H$,

(iii) there exists $\lambda \in (0,\infty)$ so that $a(x,x) \geq \lambda \|x\|^2_H$ for every $x \in H$.

Let moreover $f\colon H \to \mathbb{R}$ be a continuous linear functional. Consider the map $J\colon H \to \mathbb{R}$ given by

$$J(x) = a(x,x) - 2f(x).$$

Finally, let $K \subseteq H$ be a non-empty closed convex subset.

(a) Prove that there exists a *unique* $y_0 \in K$ such that $J(y_0) \leq J(z)$ for every $z \in K$.

(b) Prove that the unique minimizer y_0 from (a) is also the unique element of K satisfying $a(y_0, z - y_0) \geq f(z - y_0)$ for every $z \in K$.