10.1. Various notions of continuity

Suppose $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed $\mathbb{K}$-vector spaces (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$).

(a) A linear map $A\colon X \to Y$ is bounded if and only if it is $\sigma(X, X^*)$-$\sigma(Y, Y^*)$-continuous (i.e., continuous with respect to the weak topologies on X and Y).

(b) A linear map $B\colon Y^* \to X^*$ is $\sigma(Y^*, Y)$-$\sigma(X^*, X)$-continuous (i.e., continuous with respect to the weak* topologies on Y^* and X^*) if and only if there is a bounded linear operator $A\colon X \to Y$ such that $B = A^*$.

(c) A linear operator $A\colon X \to Y$ is $\sigma(X, X^*)$-$\|\cdot\|_Y$-continuous (i.e., weak–norm continuous) if and only if it is bounded and has finite rank (i.e., has finite-dimensional range).

10.2. Elementary properties of dual operators

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed $\mathbb{K}$-vector spaces (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$). Recall that if $T \in L(X, Y)$, then its dual operator T^* is in $L(Y^*, X^*)$ and it is characterised by the property

$$\langle T^* y^*, x \rangle_{X^* \times X} = \langle y^*, Tx \rangle_{Y^* \times Y} \quad \text{for every } x \in X \text{ and } y^* \in Y^*.$$

Prove the following facts about dual operators.

(a) $(\mathrm{Id}_X)^* = \mathrm{Id}_{X^*}$.

(b) If $T \in L(X, Y)$ and $S \in L(Y, Z)$, then $(S \circ T)^* = T^* \circ S^*$.

(c) If $T \in L(X, Y)$ is bijective with inverse $T^{-1} \in L(Y, X)$, then $(T^*)^{-1} = (T^{-1})^*$.

(d) Let $\mathcal{I}_X\colon X \hookrightarrow X^{**}$ and $\mathcal{I}_Y\colon Y \hookrightarrow Y^{**}$ be the canonical inclusions. Then,

$$\forall\, T \in L(X, Y): \quad \mathcal{I}_Y \circ T = (T^*)^* \circ \mathcal{I}_X.$$

10.3. Dual operators and invertibility

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed $\mathbb{K}$-vector spaces (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) and $T \in L(X, Y)$. Prove the following.

(a) If T is an isomorphism with $T^{-1} \in L(Y, X)$, then T^* is an isomorphism.

(b) If T is an isometric isomorphism, then T^* is an isometric isomorphism.

(c) If X and Y are both reflexive, then the reverse implications of (a) and (b) hold.

(d) If $(X, \|\cdot\|_X)$ is a reflexive Banach space isomorphic to the normed space $(Y, \|\cdot\|_Y)$, then Y is reflexive.

10.4. Invariant measures again

Let (K, d) be a non-empty compact metric space and let $T \in L(C(K,\mathbb{R}), C(K,\mathbb{R}))$ satisfy

- $T\mathbf{1} = \mathbf{1}$, where $\mathbf{1} := (K \ni x \mapsto 1 \in \mathbb{R}) \in C(K,\mathbb{R})$ and
- $Tf \geq 0$ for all $f \in C(K,\mathbb{R})$ with $f \geq 0$.

(a) Prove for all $n \in \mathbb{N}$ that the mapping $S_n \colon \mathcal{P}(K) \to \mathcal{P}(K)$, defined via

$$\int_K f \, d(S_n\nu) = \frac{1}{n}\sum_{k=0}^{n-1} \int_K T^k f \, d\nu \quad \text{for all } f \in C(K,\mathbb{R}), \nu \in \mathcal{P}(K),$$

is indeed well-defined.

(b) Show for all $\nu \in \mathcal{P}(K)$ that there exist $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ with $n_k \nearrow \infty$ as $k \to \infty$ and $\mu \in \mathcal{P}(K)$ such that

$$\int_K f \, d\mu = \lim_{k\to\infty} \int_K f \, d(S_{n_k}\nu) \quad \text{for all } f \in C(K,\mathbb{R}).$$

(c) Let $\nu, \mu \in \mathcal{P}(K)$ and $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ satisfy $n_k \nearrow \infty$ and $\int_K f \, d(S_{n_k}\nu) \to \int_K f \, d\mu_\infty$ as $k \to \infty$. Infer that

$$\int_K Tf \, d\mu = \int_K f \, d\mu \quad \text{for every } f \in C(K,\mathbb{R}).$$

(d) Prove for every $f \in C(K,\mathbb{R})$ with $Tf = f$ and $f \neq 0$ that there exists $\mu \in \mathcal{P}(K)$ satisfying

- $\int_K f \, d\mu \neq 0$ and
- $\int_K Tg \, d\mu = \int_K g \, d\mu$ for all $g \in C(K,\mathbb{R})$

(e) Solve Problem 9.5 (*Invariant measures à la Krylov–Bogolioubov*) again using (c) or (d).

10.5. Von Neumann's ergodic theorem

Let $(H, \langle\cdot,\cdot\rangle)$ be a $\mathbb{K}$-Hilbert space (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$), let T be a continuous linear operator on H with $\|T\|_{L(H,H)} \leq 1$, let $U := \ker(I - T)$ (with $I = (H \ni x \mapsto x \in H) \in L(H, H)$ being the identity operator), let P_U denote the orthogonal projection onto U and let $S_n := \frac{1}{n}\sum_{k=0}^{n-1} T^k$ for every $n \in \mathbb{N}$. Our goal is to show that

$$\limsup_{n\to\infty} \|S_n x - P_U x\|_H = 0 \quad \text{for all } x \in H.$$

For this, we recommend to proceed along the following steps:

(a) For all $x \in H$, we have $Tx = x$ if and only if $T^*x = x$.

(b) $U^\perp = \overline{\operatorname{im}(I - T)}$.

(c) $\lim_{n\to\infty} S_n x = x$ for all $x \in U$ and $\lim_{n\to\infty} S_n x = 0$ for all $x \in U^\perp$.

10.6. Von Neumann again

Let $(X, \|\cdot\|_X)$ be a reflexive space, let $T\colon X \to X$ be a continuous linear operator satisfying $\sup_{n\in\mathbb{N}_0}\|T^n\|_{L(X,X)} < \infty$, let $U := \ker(I - T)$ (with $I = (X \ni x \mapsto x \in X) \in L(X, X)$ being the identity operator) and let $S_n := \frac{1}{n}\sum_{k=0}^{n-1} T^k$ for every $n \in \mathbb{N}$.

(a) Prove that $Y := \{x \in X \mid \lim_{n\to\infty} S_n x \text{ exists}\}$ is a closed subspace of X.

(b) Show that $P\colon Y \to X$, defined by $Px = \lim_{n\to\infty} S_n x$ is a continuous linear map satisfying $\operatorname{im}(P) = U \subseteq Y$, $\ker(P) = \overline{\operatorname{im}(I - T)}$, and $P|_U = I|_U$. In particular, deduce that $Y = \ker(I - T) \oplus \overline{\operatorname{im}(I - T)}$.

(c) Demonstrate for every $x^* \in Y^\perp$ that $T^*x^* = x^*$ and $x^* \in U^\perp$.

(d) Show for every $x \in X$ that $U \cap \overline{\operatorname{conv}}(\{T^k x \colon k \in \mathbb{N}_0\}) \neq \emptyset$.

(e) Deduce that $Y = X$.

Hint: The reflexivity assumption is only really used in (d). For (e), use (c) and (d) to show for every $x^* \in Y^\perp$ that $x^*(x) = 0$ for every $x \in X$.