11.1. Compact operators

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces. We denote by

$$K(X,Y) = \{T \in L(X,Y) \mid \overline{T(B_1(0))} \subseteq Y \text{ compact}\}$$

the set of compact operators between X and Y. Prove the following statements.

(a) $T \in L(X,Y)$ is a compact operator if and only if every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $(Tx_{n_k})_{k\in\mathbb{N}}$ is convergent in Y.

(b) If $(Y, \|\cdot\|_Y)$ is complete, then $K(X,Y)$ is a closed subspace of $L(X,Y)$.

(c) Let $T \in L(X,Y)$. If its range $T(X) \subseteq Y$ is finite-dimensional, then $T \in K(X,Y)$.

(d) Let $T \in L(X,Y)$ and $S \in L(Y,Z)$. If T or S is a compact operator, then $S \circ T$ is a compact operator.

(e) If X is reflexive, then any operator $T \in L(X,Y)$ which maps weakly convergent sequences to strongly convergent sequences, that is

$$x_n \overset{\mathrm{w}}{\rightharpoonup} x \text{ in } X \quad \Longrightarrow \quad Tx_n \to Tx \text{ in } Y,$$

is a compact operator.

11.2. Schauder's theorem

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and let $T \in L(X,Y)$ be a bounded linear operator. Prove that T is compact if and only if T^* is compact.

Hint: To infer compactness of T^* from compactness of T, employ the Arzéla–Ascoli theorem.

11.3. Various notions of continuity – continued

Suppose $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are $\mathbb{K}$-Banach spaces (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$).

(a) Let B^* be the closed unit ball in Y^*, equipped with the weak* topology. Prove that a bounded linear operator $A\colon X \to Y$ is compact if and only if $A^*|_{B^*}\colon B^* \to X^*$ is $\sigma(B^*, Y)$-$\|\cdot\|_{X^*}$-continuous (i.e., weak*–norm continuous).

(b) Suppose $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are reflexive. A linear operator $A\colon X \to Y$ is compact if and only if $A|_B\colon B \to Y$ is $\sigma(B, X^*)$-$\|\cdot\|_Y$-continuous (i.e., weak–norm continuous).

11.4. Ehrling's lemma

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be Banach spaces, let $T \in L(X, Y)$ be compact and let $J \in L(Y, Z)$ be injective. Prove that for every $\varepsilon \in (0, \infty)$, there exists $C \in [0, \infty)$ such that

$$\|Tx\|_Y \leq \varepsilon \|x\|_X + C\|JTx\|_Z \quad \text{for all } x \in X.$$

11.5. Integral operators

Let $m \in \mathbb{N}$ and let $\emptyset \neq \Omega \subseteq \mathbb{R}^m$ be a bounded open set. Given $k \in L^2(\Omega \times \Omega, \mathbb{C})$, consider the linear operator $K\colon L^2(\Omega, \mathbb{C}) \to L^2(\Omega, \mathbb{C})$ defined by

$$(Kf)(x) = \int_\Omega k(x, y) f(y)\, dy.$$

(a) Prove that K is well-defined, i.e., $Kf \in L^2(\Omega, \mathbb{C})$ for any $f \in L^2(\Omega, \mathbb{C})$.

(b) Prove that K is a compact operator.

(c) If, in addition, the kernel k satisfies $k(x, y) = \overline{k(y, x)}$ for almost every $(x, y) \in \Omega \times \Omega$, prove that the operator $A\colon L^2(\Omega, \mathbb{C}) \to L^2(\Omega, \mathbb{C})$, defined by

$$Af = f - Kf,$$

is surjective if and only if it is injective.

11.6. Integral operators again

Let $m \in \mathbb{N}$ and let $\emptyset \neq K \subseteq \mathbb{R}^m$ be a non-empty compact set. Given $k \in C(K \times K, \mathbb{R})$, consider the linear operator $T\colon C(K, \mathbb{R}) \to C(K, \mathbb{R})$ defined by

$$(Tf)(x) = \int_K k(x, y) f(y)\, dy \quad \text{for all } f \in C(K, \mathbb{R}).$$

(a) Prove that T is well-defined, i.e., $Tf \in C(K, \mathbb{R})$ for every $f \in C(K, \mathbb{R})$.

(b) Prove that T is a compact operator.

Hint: You could approximate k by appropriate step functions and construct an approximation through finite-rank operators or you could recall the Arzéla–Ascoli theorem.

(c) If $k(x, y) = k(y, x)$ for all $x, y \in K$, prove that the operator $A\colon C(K, \mathbb{R}) \to C(K, \mathbb{R})$, defined by $Af = f - Tf$ for every $f \in C(K, \mathbb{R})$ is surjective if and only if it is injective.

Remark. In both 11.5.(c) and 11.6.(c), the symmetry assumption on k is not really necessary. Riesz–Schauder theory ensures that $\dim(\ker(I-T)) = \dim(\ker(I-T^*)) = \operatorname{codim}(\operatorname{im}(I-T)) = \operatorname{codim}(\operatorname{im}(I-T^*))$.

11.7. A dual statement

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and let $T\colon D(T) \subseteq X \to Y$ be a densely defined closed linear operator. Prove that the following properties are equivalent:

(i) T^* is surjective.

(ii) There exists $C \in [0, \infty)$ such that $\|u\|_X \leq C\|Tu\|_Y$ for all $u \in D(T)$.

(iii) T is injective and has closed range.