12.1. Spectra of shifts

Let $S\colon \ell^2(\mathbb{N},\mathbb{C}) \to \ell^2(\mathbb{N},\mathbb{C})$ be the right shift on $\ell^2(\mathbb{N},\mathbb{C})$, i.e.,

$$S((x_1, x_2, x_3, \ldots)) = (0, x_1, x_2, \ldots) \quad \text{for all } (x_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N},\mathbb{C}).$$

(a) Calculate the operator norm $\|S\|_{L(\ell^2(\mathbb{N},\mathbb{C}),\ell^2(\mathbb{N},\mathbb{C}))}$ and the spectral radius r_S of S.

(b) Determine the point spectrum $\sigma_p(S)$, the continuous spectrum $\sigma_c(S)$ and the residual spectrum $\sigma_r(S)$ of S.

(c) Do the same for S^*, the left shift.

12.2. Fredholm's alternative (on Hilbert spaces)

Let H be a Hilbert space and let $K \in L(H)$ be a compact operator. Prove the following statements. (The goal of this exercise lies in (d) and (e) below.)

(a) $\dim(\ker(I-K)) < \infty$.

(b) $\operatorname{im}(I-K)$ is closed.

(c) $\operatorname{im}(I-K) = (\ker(I-K^*))^\perp$.

(d) $\ker(I-K) = \{0\}$ if and only if $\operatorname{im}(I-K) = H$.

Hint: For "($\Rightarrow$)", assume that $\ker(I-K) = \{0\}$ and $\operatorname{im}(I-K) \neq H$. Show that this assumption leads to the following chain of proper inclusions: $H \supsetneq (I-K)(H) \supsetneq (I-K)^2(H) \supsetneq (I-K)^3(H) \supsetneq \ldots$; choose now $(x_k)_{k\in\mathbb{N}} \subseteq H$ such that $\|x_k\| = 1$, $x_k \in (I-K)^k(H)$, $x_k \in ((I-K)^{k+1}(H))^\perp$ and show that $Kx_k - Kx_l$ has norm greater or equal than 1 whenever $k < l$ because $Kx_k - Kx_l$ can be written as the difference of x_k and an element of $(I-K)^{k+1}(H)$. For "($\Leftarrow$)", dualize.

(e) $\dim(\ker(I-K)) = \dim(\ker(I-K^*))$.

Hint: Assume for a contradiction that $\dim(\ker(I-K)) < \dim(\operatorname{im}(I-K)^\perp)$. Construct an injective compact map $A_0\colon \ker(I-K) \to \operatorname{im}(I-K)^\perp$. Show that this map is not surjective. Extend A_0 to a compact map $A\colon H \to \operatorname{im}(I-K)^\perp$ with $\operatorname{im}(A) = \operatorname{im}(A_0)$ by setting $A|_{(\ker(I-K))^\perp} \equiv 0$. Show that $\ker(I-K-A) = \{0\}$, but $\operatorname{im}(I-K-A) \neq H$. This contradiction now shows $\dim(\ker(I-K)) \geq \dim(\operatorname{im}(I-K)^\perp)$. Finish by dualizing.

Remark. The statement remains true in the Banach space setting. (The proof gets slightly more technical.) In particular, we just saw – as mentioned earlier – that the extra symmetry assumption on the kernel k in Problem 11.5 (*Integral operators*) was not really necessary.

12.3. Symmetry vs. self-adjointness

Let H be a $\mathbb{C}$-Hilbert space and let $A\colon D_A \subseteq H \to H$ be a densely defined symmetric linear operator. Prove that the following statements are equivalent:

(i) A is self-adjoint.

(ii) A is closed and $\ker(A^* + i) = \{0\} = \ker(A^* - i)$.

(iii) $\operatorname{im}(A + i) = H = \operatorname{im}(A - i)$.

12.4. Special construction of self-adjoint operators

Let H and K be $\mathbb{K}$-Hilbert spaces (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) and let $J \in L(K, H)$ be an injective operator with dense range.

(a) Prove that $JJ^* \in L(H)$ is an injective operator with dense range.

(b) Prove that $S := (JJ^*)^{-1}$ (i.e., the operator $S\colon D_S \subseteq H \to H$, defined by $D_S = \operatorname{im}(JJ^*)$ and $S(JJ^*x) = x$ for all $x \in H$) is self-adjoint.

Hint: For (b), prove that S is symmetric and use Problem 12.3.

12.5. Heisenberg's Uncertainty Principle

Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space over $\mathbb{C}$. Let $D_A, D_B \subseteq H$ be dense subspaces and let $A\colon D_A \subseteq H \to H$ and $B\colon D_B \subseteq H \to H$ be symmetric linear operators. Assume that

$$A(D_A \cap D_B) \subseteq D_B \quad \text{and} \quad B(D_A \cap D_B) \subseteq D_A,$$

and define the *commutator* of A and B as

$$[A, B]\colon D_{[A,B]} \subseteq H \to H, \qquad [A, B](x) := A(Bx) - B(Ax),$$

where $D_{[A,B]} := D_A \cap D_B$.

(a) Prove that

$$\Big|\langle x, [A, B]x\rangle_H\Big| \leq 2\|Ax\|_H \|Bx\|_H \quad \text{for every } x \in D_{[A,B]}.$$

(b) Define now the *standard deviation* of A

$$\varsigma(A,x) := \sqrt{\langle Ax, Ax\rangle_H - \langle x, Ax\rangle_H^2}$$

at each $x \in D_A$ with $\|x\|_H = 1$. Verify that $\varsigma(A,x)$ is well-defined for every x (i.e. that the radicand is real and non-negative) and prove that for every $x \in D_{[A,B]}$ with $\|x\|_H = 1$ there holds

$$\Big|\langle x, [A,B]x\rangle_H\Big| \leq 2\varsigma(A,x)\,\varsigma(B,x).$$

Remark. The possible *states* of a quantum mechanical system are given by elements $x \in H$ with $\|x\|_H = 1$. Each *observable* is given by a symmetric linear operator $A\colon D_A \subseteq H \to H$. If the system is in state $x \in D_A$, we measure the observable A with uncertainty $\varsigma(A,x)$.

(c) Let $A\colon D_A \subseteq H \to H$ and $B\colon D_B \subseteq H \to H$ be as above. A, B is called *Heisenberg pair* if

$$[A,B] = i\,\mathrm{Id}\,|_{D_{[A,B]}}.$$

Show that, if A, B is a Heisenberg pair with B continuous (and $D_B = H$), then A cannot be continuous.

(d) Consider the Hilbert space $(H, \langle\cdot,\cdot\rangle_H) = \Big(L^2([0,1],\mathbb{C}), \langle\cdot,\cdot\rangle_{L^2}\Big)$ and the subspace

$$C^1_0([0,1],\mathbb{C}) := \{f \in C^1([0,1],\mathbb{C}) \mid f(0) = 0 = f(1)\}.$$

Recall that $C^1_0([0,1],\mathbb{C}) \subseteq L^2([0,1],\mathbb{C})$ is a dense subspace. The operators

$$P\colon C^1_0([0,1],\mathbb{C}) \to L^2([0,1],\mathbb{C}), \quad f(s) \mapsto if'(s) \qquad\qquad Q\colon L^2([0,1],\mathbb{C}) \to L^2([0,1],\mathbb{C}), \quad f(s) \mapsto sf(s)$$

correspond to the observables *momentum* and *position*. Check that P and Q are well-defined, symmetric operators. Check that $[P,Q]\colon C^1_0([0,1],\mathbb{C}) \to L^2([0,1],\mathbb{C})$ is well-defined.

Show that P and Q form a Heisenberg pair and conclude that the *uncertainty principle* holds: for every $f \in C^1_0([0,1],\mathbb{C})$ with $\|f\|_{L^2([0,1],\mathbb{C})} = 1$ there holds

$$\varsigma(P,f)\,\varsigma(Q,f) \geq \tfrac{1}{2}.$$

Thus we conclude: *The more precisely the momentum of some particle is known, the less precisely its position can be known, and vice versa.*