13.1. Friedrich extension

Let $(H, \langle\cdot,\cdot\rangle_H)$ be a $\mathbb{K}$-Hilbert space (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) and let $A\colon D_A \subseteq H \to H$ be a densely defined linear operator satisfying that

- A is *symmetric*, i.e., $\forall x, y \in D_A\colon \langle Ax, y\rangle_H = \langle x, Ay\rangle_H$ and
- A is *bounded below*, i.e., there exists $C \in \mathbb{R}$ such that $\langle Ax, x\rangle_H \geq C\|x\|_H^2$ for all $x \in D_A$.

Our goal is to show that A possesses a self-adjoint extension B (i.e., $A \subseteq B = B^*$) with $\langle Bx, x\rangle_H \geq C\|x\|_H^2$ for all $x \in D_B$.

(a) Find $\lambda \in \mathbb{R}$, $\varepsilon \in (0,\infty)$ so that $a\colon D_A \times D_A \ni (x,y) \mapsto \langle Ax + \lambda x, y\rangle_H \in \mathbb{K}$ defines an inner product on D_A which satisfies for all $x \in D_A$ that $a(x,x) \geq \varepsilon\|x\|_H^2$.

(b) Consider the metric space (D_A, d_A) where $d_A(x,y) := \sqrt{a(x-y, x-y)}$ for all $x, y \in D_A$ with a as in (a). Let (K, d_K, ι) be a completion of (D_A, d_A) (cp. Problem 3.3 (*Completion of metric spaces*)). Prove that there exists a unique vector space structure on K so that ι is linear and the vector space operations $K \times K \ni (x,y) \mapsto x + y \in K$ and $\mathbb{K} \times K \ni (\mu, x) \mapsto \mu x \in K$ are continuous (w.r.t. the obvious choices of topologies). In addition, show that there even exists a unique scalar product $\langle\cdot,\cdot\rangle_K\colon K \times K \to \mathbb{K}$ such that for all $x, y \in K$ it holds that $d_K(x,y) = \sqrt{\langle x-y, x-y\rangle_K}$.

(c) With (K, d_K, ι) being a completion of (D_A, d_A), equipped with the Hilbert space structure (in particular, with the scalar product $\langle\cdot,\cdot\rangle_K\colon K \times K \to \mathbb{K}$) shown to exist in (b), argue that there exists an injective bounded linear map $J\colon K \to H$ satisfying for all $x \in D_A$, $y \in K$ that

$$\langle \iota(x), y\rangle_K = \langle Ax + \lambda x, Jy\rangle_H.$$

Moreover, prove that $J(K) \subseteq H$ can be written as

$$J(K) = \Big\{ x_\infty \in H \mid \exists (x_n)_{n\in\mathbb{N}} \subseteq D_A \text{ with } \limsup_{n\to\infty} \|x_n - x_\infty\|_H = 0 \text{ and } \limsup_{N\to\infty} \sup_{m,n\geq N} a(x_n - x_m, x_n - x_m) = 0 \Big\}.$$

(d) Show that operator $B\colon D_B \subseteq H \to H$, defined by $D_B := \operatorname{im}(JJ^*)$, $B(JJ^*u) := u - \lambda JJ^*u$ for all $u \in H$, is well-defined and a self-adjoint extension of A (i.e., $D_A \subseteq D_B$) with $\langle Bx, x\rangle_H \geq C\|x\|_H^2$ for all $x \in D_B$.

13.2. The Dirichlet–Laplace operator as a Friedrich extension

Let $A\colon C_c^\infty((0,1),\mathbb{R}) \subseteq L^2((0,1),\mathbb{R}) \to L^2((0,1),\mathbb{R})$ be defined by $Af = -f''$ for all $f \in C_c^\infty((0,1),\mathbb{R})$. Our goal is to construct the Friedrich extension of A.

(a) Prove that $(C_c^\infty((0,1),\mathbb{R}), a)$ with a defined via $a(u,v) = \int_{(0,1)} u'v'\,dx$ for all $u, v \in C_c^\infty((0,1),\mathbb{R})$ is an inner product space and prove that there exists $c \in (0,\infty)$ such that for all $u \in C_c^\infty((0,1),\mathbb{R})$ it holds that $\int_0^1 |u|^2\,dx \leq c \int_0^1 |u'|^2\,dx$.

(b) Let $K \subseteq L^2((0,1),\mathbb{R})$ be such that $u \in K$ if and only if there exists a sequence $(f_n)_{n\in\mathbb{N}} \subseteq C_c^\infty((0,1),\mathbb{R})$ such that

- $f_n \to u$ in $L^2((0,1),\mathbb{R})$ as $n \to \infty$ and
- $(f_n')_{n\in\mathbb{N}} \subseteq L^2((0,1),\mathbb{R})$ is a Cauchy sequence.

Prove that, for every $u \in K$, there exists a unique $w \in L^2((0,1),\mathbb{R})$ such that

$$\int_{(0,1)} w\varphi\,dx = -\int_{(0,1)} u\varphi'\,dx \quad \text{for all } \varphi \in C_c^\infty((0,1),\mathbb{R}).$$

Afterwards, we shall always write $w = u'$ in such a situation (as w equals the classical derivative in the case of smooth functions).

(c) Prove that $\langle\cdot,\cdot\rangle_K\colon K \times K \ni (u,v) \mapsto \int_{(0,1)} u'v'\,dx \in \mathbb{R}$ defines a scalar product on K and that $(K, \langle\cdot,\cdot\rangle_K)$ is a completion of $(C_c^\infty((0,1),\mathbb{R}), a)$.

(d) Prove that the Friedrich extension B as in Problem 13.1 (*Friedrich extension*) is given as follows: $u \in D_B$ if and only if $u \in K$ and there exists $g \in L^2((0,1),\mathbb{R})$ such that for all $\varphi \in C_c^\infty((0,1),\mathbb{R})$ it holds that $\int_0^1 u\varphi''\,dx = \int_0^1 g\varphi\,dx$; in this case, $g = Bu$.

(e) Prove that the embedding $J\colon (K, \|\cdot\|_K) \ni f \mapsto f \in (L^2(0,1),\mathbb{R}), \|\cdot\|_{L^2})$ is compact. In addition, prove that every element of K has a unique continuous representative and that this continuous representative extends uniquely to a continuous function on $[0,1]$ vanishing on $\{0,1\}$.

(f) Infer that $B^{-1}\colon L^2((0,1),\mathbb{R}) \to L^2((0,1),\mathbb{R})$ is a compact operator.

(g) Determine the spectrum of B as well as an orthonormal basis of $L^2((0,1),\mathbb{R})$ consisting of eigenvectors of B (respectively of B^{-1}).

(h) Express B (and, especially, D_B) and B^{-1} with the help of these eigenvalues and eigenvectors. Can you find a way to define B^s for $s \in (0,\infty)$?

13.3. Spectral properties of generators of C^0-semigroups

Let $(X, \|\cdot\|_X)$ be a Banach space and let $T = (T_t)_{t\in[0,\infty)} \subseteq L(X)$ be a C^0-semigroup, that is,

- $T_0 = I$,
- $T_{t+s} = T_t T_s$ for all $t, s \in [0, \infty)$, and
- $\limsup_{t\searrow 0} \|T_t x - x\|_X = 0$ for all $x \in X$.

We know that there exist $M \in [1, \infty)$, $\omega \in \mathbb{R}$ such that $\|T_t\|_{L(X)} \leq Me^{\omega t}$ for all $t \in [0, \infty)$. The *generator* of T is the operator $A\colon D_A \subseteq X \to X$ defined by

$$D_A := \left\{ x \in X \mid \lim_{t\searrow 0} \frac{T_t x - x}{t} \text{ exists} \right\} \quad \text{and} \quad Ax := \lim_{t\searrow 0} \frac{T_t x - x}{t} \quad \text{for all } x \in D_A.$$

(a) Prove for all $t \in [0, \infty)$, $x \in D_A$ that $T_t x \in D_A$ and $AT_t x = T_t Ax$.

(b) Show for all $t \in [0, \infty)$, $x \in X$ that $\int_0^t T_s x\, ds \in D_A$ and $A(\int_0^t T_s x\, ds) = T_t x - x$.

(c) Show for all $t \in [0, \infty)$, $x \in D_A$ that $\int_0^t T_s Ax\, ds = A \int_0^t T_s x\, ds$.

(d) Prove for all $x \in D_A$ that the function $u := ([0, \infty) \ni t \mapsto T_t x \in X)$ satisfies that $u \in C^1([0, \infty), (X, \|\cdot\|_X)) \cap C([0, \infty), (D_A, \|\cdot\|_{D_A}))$ and that $u'(t) = Au(t)$ for all $t \in [0, \infty)$.

(e) Demonstrate that A is densely defined and closed.

(f) Prove for all $\lambda \in (\omega, \infty)$ that $R_\lambda \in L(X)$, given by $R_\lambda x = \int_0^\infty e^{-\lambda t} T_t x\, dt$ for all $x \in X$, is well-defined and satisfies

- $R_\lambda(\lambda - A)x = x$ for all $x \in D_A$,
- $R_\lambda x \in D_A$ and $(\lambda - A)R_\lambda x = x$ for all $x \in X$.

(g) Conclude for all $\lambda \in (\omega, \infty)$ that $\lambda \in \varrho(A)$, i.e., $\lambda - A$ is continuously invertible, with

$$\|(\lambda - A)^{-n}\|_{L(X)} \leq \frac{M}{(\lambda - \omega)^n} \quad \text{for all } n \in \mathbb{N}.$$

Remark. The Hille–Yosida theorem, in fact, ensures that any densely defined closed operator A on a Banach space X which satisfies for some $M, \omega \in \mathbb{R}$ that $(\omega, \infty) \subseteq \varrho(A)$ and $\|(\lambda - A)^{-n}\|_{L(X)} \leq \frac{M}{(\lambda-\omega)^n}$ for all $\lambda \in (\omega, \infty)$, $n \in \mathbb{N}$, is the generator of a C^0-semigroup.

13.4. A heat semigroup

Let $B\colon D_B \subseteq L^2((0,1),\mathbb{R}) \to L^2((0,1),\mathbb{R})$ denote the self-adjoint extension of $A\colon C_c^\infty((0,1),\mathbb{R}) \subseteq L^2((0,1),\mathbb{R}) \to L^2((0,1),\mathbb{R}))$, given by $Af = -f''$ for all $f \in D_A$, which we constructed in Problem 13.2 (*The Dirichlet–Laplace operator as a Friedrich extension*). In this exercise we dwell on the spectral representation of B obtained in part (h) of Problem 13.2 to construct the associated C^0-semigroup and prove some regularity properties.

(a) Prove that there exists a C^0-semigroup $(T_t)_{t\in[0,\infty)} \subseteq L(L^2((0,1),\mathbb{R}))$ whose generator is $-B$.

(b) Prove for all $t \in (0,\infty)$, $f \in L^2((0,1),\mathbb{R})$ that

- $T_t f \in C^\infty([0,1],\mathbb{R})$ in the sense that there is a (necessarily unique) element of $C^\infty([0,1],\mathbb{R})$ in the L^2-equivalence class $T_t f$ and
- $(T_t f)(0) = 0 = (T_t f)(1)$ in the sense that the unique element of $C^\infty([0,1],\mathbb{R})$ in the L^2-equivalence class $T_t f$ takes on the value 0 on $\{0,1\}$.

(c) Prove for all $x \in (0,1)$, $f \in L^2((0,1),\mathbb{R})$ that $(0,\infty) \ni t \mapsto (T_t f)(x) \in \mathbb{R}$ is smooth (where we identify $T_t f \in L^2((0,1),\mathbb{R})$ for $t \in (0,\infty)$, $f \in L^2((0,1),\mathbb{R})$ with its continuous representative, which exists according to (b)) so that $(T_t f)(x)$ is defined for every $x \in (0,1)$.

(d) Prove for all $f \in L^2((0,1),\mathbb{R})$ that $u\colon (0,\infty)\times[0,1] \to \mathbb{R}$, given by $u(t,x) = (T_t f)(x) \in \mathbb{R}$ for all $t \in (0,\infty)$, $x \in [0,1]$, satisfies that

$$\begin{cases} (\partial_t u)(t,x) &= (\partial_x^2 u)(t,x), \qquad \text{for all } t\in(0,\infty), x\in(0,1),\\ u(t,0) &= 0, \qquad \text{for all } t\in(0,\infty),\\ u(t,1) &= 0, \qquad \text{for all } t\in(0,\infty),\\ \limsup_{t\searrow 0}\|u(t,\cdot)-f\|_{L^2} &= 0. \end{cases}$$

(e) Finally, prove for all $f \in L^2((0,1),\mathbb{R})$, $v \in C^\infty((0,\infty)\times[0,1],\mathbb{R})$ satisfying

$$\begin{cases} (\partial_t v)(t,x) &= (\partial_x^2 v)(t,x) \qquad \text{for all } t\in(0,\infty), x\in(0,1),\\ v(t,0) &= 0 \qquad \text{for all } t\in(0,\infty),\\ v(t,1) &= 0 \qquad \text{for all } t\in(0,\infty),\\ \limsup_{t\searrow 0}\|v(t,\cdot)-f\|_{L^2} &= 0, \end{cases}$$

that $v(t,x) = (T_t f)(x)$ for all $(t,x) \in (0,\infty)\times[0,1]$.