1.1. Completeness, closedness, compactness, and metric spaces

(a) If $Y \subseteq X$ is a complete subspace (i.e., $Y \subseteq X$ and $(Y, d|_{Y\times Y})$ is complete), then Y is closed (i.e., a closed subset of X).

Solution: Let $(y_n)_{n\in\mathbb{N}} \subseteq Y$ and $x_0 \in X$ satisfy that $\limsup_{n\to\infty} d(y_n, x_0) = 0$ (i.e., $(y_n)_{n\in\mathbb{N}}$ is a sequence in Y which converges in (X, d)). For the proof it suffices to show that $x_0 \in Y$. The assumption that $(y_n)_{n\in\mathbb{N}}$ is a converging sequence in (X, d) implies that $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (X, d) as well as in $(Y, d|_{Y\times Y})$. The assumption that $(Y, d|_{Y\times Y})$ is complete ensures that there exists $y_0 \in Y$ such that $\limsup_{n\to\infty} d(y_n, y_0) = \limsup_{n\to\infty} d|_{Y\times Y}(y_n, y_0) = 0$. Hence, we obtain that

$$d(x_0, y_0) \le \limsup_{n\to\infty}(d(x_0, y_n) + d(y_n, y_0)) \le \limsup_{n\to\infty} d(x_0, y_n) + \limsup_{n\to\infty} d(y_n, y_0) = 0,$$

which implies that $x_0 = y_0 \in Y$, as desired.

(b) If (X, d) is complete, then every closed subset $Y \subseteq X$ is complete (i.e., $(Y, d|_{Y\times Y})$ is complete).

Solution: Let $(y_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(Y, d|_{Y\times Y})$. This implies that $(y_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence in (X, d). By the completeness of (X, d), there exists $x_0 \in X$ such that $\limsup_{n\to\infty} d(y_n, x_0) = 0$. The closedness of Y in (X, d) ensures that $x_0 \in Y$. Hence, we have that

$$\limsup_{n\to\infty} d|_{Y\times Y}(y_n, x_0) = \limsup_{n\to\infty} d(y_n, x_0) = 0,$$

that is, $(y_n)_{n\in\mathbb{N}}$ converges to y_0 in $(Y, d|_{Y\times Y})$.

(c) If (X, d) is compact, then (X, d) is complete.

Solution: Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (X, d). The compactness of (X, d) implies that there exists $x_0 \in X$ and a sequence $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ with $n_k \nearrow \infty$ for $k \to \infty$ such that $\limsup_{k\to\infty} d(x_{n_k}, x_0) = 0$. Hence:

$$\begin{aligned}
\limsup_{n\to\infty} d(x_n, x_0) &= \limsup_{n\to\infty} \sup_{m\ge n} d(x_m, x_0) \\
&= \limsup_{k\to\infty} \sup_{m\ge n_k} d(x_m, x_0) \\
&\le \limsup_{k\to\infty} \sup_{m\ge n_k} (d(x_m, x_{n_k}) + d(x_{n_k}, x_0)) \\
&\le \limsup_{k\to\infty} \sup_{m\ge n_k} d(x_m, x_{n_k}) + \limsup_{k\to\infty} d(x_{n_k}, x_0) = 0,
\end{aligned}$$

that is, $(x_n)_{n\in\mathbb{N}}$ converges to x_0. This proves that (X, d) is complete.

1.2. Metrics on sequence spaces

Let (M, d) be a metric space. Consider the set of all M-valued sequences

$$S = \{(s_n)_{n\in\mathbb{N}} \mid \forall n \in \mathbb{N}\colon s_n \in M\}.$$

Let the function $\delta\colon S \times S \to [0, \infty)$ be defined by

$$\delta((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} 2^{-n} \frac{d(x_n, y_n)}{1 + d(x_n, y_n)}.$$

(a) Show that δ is a metric on S.

Solution: Note first that for all $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in S$ we have that:

$$\delta((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) = \sum_{n\in\mathbb{N}} 2^{-n} \frac{d(x_n, y_n)}{1 + d(x_n, y_n)} \in \left[0, \sum_{n\in\mathbb{N}} 2^{-n}\right] \subseteq [0, \infty).$$

So δ is well-defined. Clearly, δ is symmetric and vanishes if and only if it holds for every $n \in \mathbb{N}$ that $d(x_n, y_n) = 0$ (and hence $x_n = y_n$), which is equivalent to $(x_n)_{n\in\mathbb{N}} = (y_n)_{n\in\mathbb{N}}$ in S. Next we prove that the triangle inequality holds. Let $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}} \in S$. Note that for every $n \in \mathbb{N}$ there holds:

$$\begin{aligned}
\frac{d(x_n, z_n)}{1 + d(x_n, z_n)} &= 1 - \frac{1}{1 + d(x_n, z_n)} \\
&\leq 1 - \frac{1}{1 + d(x_n, y_n) + d(y_n, z_n)} \\
&= \frac{d(x_n, y_n) + d(y_n, z_n)}{1 + d(x_n, y_n) + d(y_n, z_n)} \\
&= \frac{d(x_n, y_n)}{1 + d(x_n, y_n) + d(y_n, z_n)} + \frac{d(y_n, z_n)}{1 + d(x_n, y_n) + d(y_n, z_n)} \\
&\leq \frac{d(x_n, y_n)}{1 + d(x_n, y_n)} + \frac{d(y_n, z_n)}{1 + d(y_n, z_n)}.
\end{aligned}$$

Summation implies:

$$\delta((x_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}}) \leq \delta((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) + \delta((y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}}).$$

That is, the triangle inequality holds. Thus, δ is a metric on S.

(b) Prove that (S, δ) is a complete metric space if (M, d) is a complete metric space.

Solution: Note first that a Cauchy sequence in S is a sequence $(a_k)_{k\in\mathbb{N}}$ of sequences $a_k = (a_{k,n})_{n\in\mathbb{N}}$, $k \in \mathbb{N}$, so that, for every $\varepsilon \in (0,\infty)$, there exists $K \in \mathbb{N}$ so that

$$\delta(a_k, a_l) = \sum_{n\in\mathbb{N}} 2^{-n} \frac{d(a_{k,n}, a_{l,n})}{1 + d(a_{k,n}, a_{l,n})} < \varepsilon \quad \text{for all } k, l \geq K.$$

Claim: *$(a_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in (S, δ) if and only if, for every fixed $n \in \mathbb{N}$, $(a_{k,n})_{k\in\mathbb{N}}$ is a Cauchy sequence in (M, d).*

Proof of the Claim: Sufficiency: let $(a_k)_{k\in\mathbb{N}}$ be Cauchy in (S, δ). Then there exists $K\colon (0,\infty) \to \mathbb{N}$ such that for all $\varepsilon \in (0,\infty)$, $k, l \in \mathbb{N}$ with $k, l \geq K_\varepsilon$ it holds that

$$\delta(a_k, a_l) = \sum_{m\in\mathbb{N}} 2^{-m} \frac{d(a_{k,m}, a_{l,m})}{1 + d(a_{k,m}, a_{l,m})} \leq \varepsilon.$$

Consequently, it follows for all $\varepsilon \in (0,\infty)$, $n, k, l \in \mathbb{N}$ with $k, l \geq K_\varepsilon$ that

$$\frac{d(a_{k,n}, a_{l,n})}{1 + d(a_{k,n}, a_{l,n})} \leq 2^n \varepsilon.$$

Thus, for all $\varepsilon \in (0,\infty)$, $n, k, l \in \mathbb{N}$ with $k, l \geq K_{\min\{\varepsilon, 2^{-(n+1)}\}}$ we deduce

$$d(a_{k,n}, a_{l,n}) \leq \frac{2^n \varepsilon}{1 - 2^n \varepsilon}.$$

This implies for every $n \in \mathbb{N}$ that $(a_{k,n})_{k\in\mathbb{N}} \subseteq M$ is Cauchy in (M, d).

Necessity: assume for every $n \in \mathbb{N}$ that $(a_{k,n})_{k\in\mathbb{N}}$ is Cauchy in (M, d). Note that, for every $k, l, N \in \mathbb{N}$ we may always estimate

$$\begin{aligned}\sum_{n\in\mathbb{N}} 2^{-n} \frac{d(a_{k,n}, a_{l,n})}{1 + d(a_{k,n}, a_{l,n})} &= \sum_{n=0}^{N} 2^{-n} \frac{d(a_{k,n}, a_{l,n})}{1 + d(a_{k,n}, a_{l,n})} + \sum_{n\geq N+1} 2^{-n} \frac{d(a_{k,n}, a_{l,n})}{1 + d(a_{k,n}, a_{l,n})} \\ &\leq \sum_{n=0}^{N} 2^{-n} \frac{d(a_{k,n}, a_{l,n})}{1 + d(a_{k,n}, a_{l,n})} + 2^{-N}.\end{aligned}$$

By assumption, for every $\varepsilon \in (0,\infty)$ and every $n \in \mathbb{N}$, there exists $N(\varepsilon, n) \in \mathbb{N}$ so that $d(a_{n,k}, a_{n,l}) \leq \varepsilon$ for all $k, l \geq N(\varepsilon, n)$. Moreover, for every $\varepsilon \in (0,\infty)$ there exists $M_\varepsilon \in \mathbb{N}$ with $2^{-M_\varepsilon} \leq \varepsilon$. Finally, define $K\colon (0,\infty) \to \mathbb{N}$ by $K_\varepsilon = \max\{M_\varepsilon, N(\varepsilon, 1), \ldots, N(\varepsilon, M_\varepsilon)\}$ for every $\varepsilon \in (0,\infty)$. Consequently, we obtain for all $\varepsilon \in (0,\infty)$, $k, l \in \mathbb{N}$ with $k, l \geq K_\varepsilon$ that

$$\delta(a_k, a_l) = \sum_{n=0}^{M_\varepsilon} 2^{-n} \frac{d(a_{k,n}, a_{l,n})}{1 + d(a_{k,n}, a_{l,n})} + 2^{-M_\varepsilon} \leq \sum_{n=0}^{M_\varepsilon} 2^{-n} \varepsilon + \varepsilon \leq 3\varepsilon,$$

which implies that $(a_k)_{k\in\mathbb{N}}$ is Cauchy in (S, δ). □

Let now $(a_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in S. By the Claim, for every $n \in \mathbb{N}$, $(a_{k,n})_{k\in\mathbb{N}}$ is Cauchy in (M, d) and thus converges to some element $\alpha_n \in M$. Defining $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ as an element of S and arguing as in the proof of the Claim above ("necessity" part), it is possible to find, for every $\varepsilon \in (0, \infty)$, some $K_\varepsilon \in \mathbb{N}$ so that $d(a_k, a) \leq \varepsilon$ for every $k \geq K_\varepsilon$. Consequently $(a_k)_{k\in\mathbb{N}}$ converges to α in (S, d). This establishes that (S, δ) is complete.

1.3. Bounded metrics

Let (X, d) be a metric space and let $\mathcal{T}$ be the topology on X which is induced by d. Prove that there exists a metric δ on X which induces the same topology $\mathcal{T}$ and is bounded, i.e., there exists $C \in \mathbb{R}$ such that for all $x, y \in X$ it holds that $\delta(x, y) \leq C$.

Solution: Let $\delta\colon X \times X \to [0, \infty)$ satisfy for all $x, y \in X$ that

$$\delta(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Calculations similar to the ones in exercise 1.2 demonstrate that δ is a metric on X. Moreover, it holds clearly for every $x, y \in X$ that $\delta(x, y) \leq 1$. It remains to show that δ induces $\mathcal{T}$. Note that for all $\varepsilon \in (0, 1)$, $x \in X$ it holds that

$$\{y \in X : \delta(y, x) < \varepsilon\} = \left\{y \in X : d(y, x) < \frac{\varepsilon}{1 - \varepsilon}\right\}.$$

This and the fact that $\lim_{\varepsilon\to 0} \frac{\varepsilon}{1-\varepsilon} = 0$ imply that every open set w.r.t. d is also open w.r.t. δ and vice versa.

1.4. Cantor's intersection theorem

The diameter of a subset A of a metric space (X, d) is defined by

$$\operatorname{diam}(A) = \sup(\{0\} \cup \{d(x, y) \mid x, y \in A\}).$$

(a) Prove that a metric space (X, d) is complete if and only if it holds for every nested sequence $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ of non-empty closed subsets $A_n \subseteq X$, $n \in \mathbb{N}$, with $\operatorname{diam}(A_n) \to 0$ for $n \to \infty$ that $\bigcap_{n\in\mathbb{N}} A_n \neq 0$. Moreover, prove that in this case $\bigcap_{n\in\mathbb{N}} A_n$ has exactly one element.

Solution: First, let us assume that (X, d) is complete and let $A_1 \supseteq A_2 \supseteq \ldots$ be a nested sequence of non-empty closed subsets satisfying $\limsup_{n\to\infty} \operatorname{diam}(A_n) = 0$. Since for every $n \in \mathbb{N}$ it is assumed that $A_n \neq \emptyset$, there exists a sequence $(a_n)_{n\in\mathbb{N}} \subseteq X$ so that for every $n \in \mathbb{N}$ it holds that $a_n \in A_n$. Note that the nestedness of the sets A_n, $n \in \mathbb{N}$, implies for all $m, n \in \mathbb{N}$ with $n > m$ that $a_n \in A_n \subseteq A_m$. Combining this with the assumption on the diameters of the sets A_n, $n \in \mathbb{N}$, we obtain for all $N \in \mathbb{N}$ that

$$\limsup_{N\to\infty} \sup_{m,n\geq N} d(a_m, a_n) \leq \limsup_{N\to\infty} \operatorname{diam}(A_N) = 0.$$

Hence, $(a_n)_{n\in\mathbb{N}}$ is Cauchy. The assumed completeness of (X, d) ensures that the Cauchy sequence $(a_n)_{n\in\mathbb{N}}$ converges to some $a_0 \in X$. Finally, observing that for all $n \in \mathbb{N}$ it holds that $\{a_m : m \geq n\} \subseteq A_n$, we find – using the closedness of the sets A_m, $m \in \mathbb{N}$ – for every $n \in \mathbb{N}$ that $a = \lim_{m\to\infty} a_m \in A_n$. Thus, $a \in \bigcap_{n\in\mathbb{N}} A_n$. Moreover, note that for all $b \in \bigcap_{n\in\mathbb{N}} A_n$ we get, due to the fact that for every $n \in \mathbb{N}$ it holds that $a, b \in A_n$:

$$d(a, b) = \limsup_{n\to\infty} d(a, b) \leq \limsup_{n\to\infty} \operatorname{diam}(A_n) = 0.$$

That is, $a = b$ and $\bigcap_{n\in\mathbb{N}} A_n = \{a\}$.

Next we prove the converse. Assume that the metric space (X, d) has the property that for every nested sequence $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ with $\operatorname{diam}(A_n) \to 0$ as $n \to \infty$ it holds that $\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset$. Let $(a_n)_{n\in\mathbb{N}} \subseteq X$ be a Cauchy sequence. Due to this, there exists a sequence $(N_k)_{k\in\mathbb{N}}$ with $N_1 < N_2 < N_3 < \ldots$ such that for all $k \in \mathbb{N}$, $m, n \geq N_k$ it holds that $d(a_m, a_n) < 2^{-k}$. Finally, let for every $k \in \mathbb{N}$ the set A_k be given by

$$A_k = \{x \in X : d(x, a_{N_k}) \leq 2^{1-k}\}.$$

First we note that for every $k \in \mathbb{N}$ it holds that A_k is closed. Moreover, we have for every $k \in \mathbb{N}$ that $\limsup_{k\to\infty} \operatorname{diam}(A_k) \leq \limsup_{k\to\infty} 2^{2-k} = 0$. Furthermore, note that for every $k \in \mathbb{N}$ and every $x \in A_{k+1}$ it holds that

$$d(x, a_{N_k}) \leq d(x, a_{N_{k+1}}) + d(a_{N_{k+1}}, a_{N_k}) \leq 2^{1-(k+1)} + 2^{-k} = 2^{1-k}.$$

Thus, for every $k \in \mathbb{N}$ we have $A_k \supseteq A_{k+1}$. According to the assumption, there exists $a_\infty \in X$ satisfying $a_\infty \in A_k$ for every $k \in \mathbb{N}$. This implies that

$$\begin{aligned}\limsup_{n\to\infty} d(a_n, a_\infty) &= \limsup_{k\to\infty} \sup_{n\geq N_k} d(a_n, a_\infty) \leq \limsup_{k\to\infty} (d(a_n, a_{N_k}) + d(a_{N_k}, a_\infty)) \\ &\leq \limsup_{k\to\infty} (2^{-k} + \operatorname{diam}(A_k)) = 0.\end{aligned}$$

Thus, $(a_n)_{n\in\mathbb{N}}$ has a limit in X. This completes the proof that (X, d) is complete.

(b) Find an example of a complete metric space and a nested sequence of non-empty closed bounded subsets with empty intersection.

Solution: Let $X := l^\infty := \{(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R} \mid \sup_{n\in\mathbb{N}} |x_n| < \infty\}$ be the space of bounded real-valued sequences. Note that $\|\cdot\|_\infty \colon X \to [0,\infty)$, defined by $\|(x_n)_{n\in\mathbb{N}}\|_\infty = \sup_{n\in\mathbb{N}} |x_n|$, defines a norm on l^∞. Let d denote the metric induced by $\|\cdot\|_\infty$. Note that (X,d) is complete (why?). Finally, let for every $n\in\mathbb{N}$ the sequence $e_n = (e_{n,k})_{k\in\mathbb{N}} \in X$ be defined via

$$e_{n,k} = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{else.} \end{cases}$$

Note that for all $n, m \in \mathbb{N}$ it holds that

$$d(e_n, e_m) = \begin{cases} 0 & \text{if } n = m, \\ 1 & \text{else.} \end{cases}$$

The sets $A_n = \{e_m \mid m \geq n\}$, $n\in\mathbb{N}$, are therefore closed, nested, non-empty, and bounded. But their intersection is empty as for every $n \in \mathbb{N}$ we have that $e_n \notin A_{n+1}$.

1.5. Intrinsic Characterisations

Let V be a vector space over $\mathbb{R}$. Prove the following equivalences.

(a) The norm $\|\cdot\|$ is induced by a scalar product $\langle\cdot,\cdot\rangle$ (in the sense that there exists a scalar product $\langle\cdot,\cdot\rangle$ such that $\forall x \in V: \ \|x\|^2 = \langle x, x\rangle$)

$\Leftrightarrow$ the norm satisfies the *parallelogram identity*, i. e. $\forall x, y \in V$:

$$\|x+y\|^2 + \|x-y\|^2 = 2\Big(\|x\|^2 + \|y\|^2\Big).$$

Hint. If $\|\cdot\|$ satisfies the parallelogram identity, consider $\langle x, y\rangle := \frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2$. Prove $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$ first for $\lambda\in\mathbb{N}$, then for $\lambda\in\mathbb{Q}$ and finally for $\lambda\in\mathbb{R}$.

Solution: If the norm $\|\cdot\|$ is induced by the scalar product $\langle\cdot,\cdot\rangle$, then the parallelogram identity holds:

$$\begin{aligned} &\|x+y\|^2 + \|x-y\|^2 \\ &= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\ &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle + \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle \\ &= 2\|x\|^2 + 2\|y\|^2. \end{aligned}$$

Conversely, we assume that $\|\cdot\|$ satisfies the parallelogram identity and claim that

$$\langle x, y\rangle := \tfrac{1}{4}\|x+y\|^2 - \tfrac{1}{4}\|x-y\|^2$$

defines a scalar product which induces $\|\cdot\|$.

- *Symmetry.* Since $\|x-y\| = \|(-1)(y-x)\| = \|y-x\|$ and since $x+y=y+x$, we have $\langle x,y\rangle = \langle y,x\rangle$ for all $x,y\in V$.

- *Linearity.* Let $x,y,z\in V$. We use the parallelogram identity in the following way.

$$\|(x+z)+y\|^2 + \|(x+z)-y\|^2 = 2\|x+z\|^2 + 2\|y\|^2.$$

We rewrite the equation above to obtain

$$\|x+y+z\|^2 = 2\|x+z\|^2 + 2\|y\|^2 - \|x-y+z\|^2 =: A$$

and switch the roles of x and y to get

$$\|x+y+z\|^2 = 2\|y+z\|^2 + 2\|x\|^2 - \|y-x+z\|^2 =: B.$$

Therefore,

$$\begin{aligned}\|x+y+z\|^2 &= \frac{A}{2} + \frac{B}{2}\\ &= \|x+z\|^2 + \|y\|^2 + \|y+z\|^2 + \|x\|^2 - \frac{\|x-y+z\|^2 + \|y-x+z\|^2}{2}.\end{aligned} \tag{1}$$

Analogously,

$$\begin{aligned}&\|x+y-z\|^2\\ &= \|x-z\|^2 + \|y\|^2 + \|y-z\|^2 + \|x\|^2 - \frac{\|x-y-z\|^2 + \|y-x-z\|^2}{2}.\end{aligned} \tag{2}$$

Note that the last term of (1) agrees with the last term of (2). Hence, we have

$$\begin{aligned}\langle x+y,z\rangle &= \tfrac14\|x+y+z\|^2 - \tfrac14\|x+y-z\|^2\\ &= \tfrac14\Big(\|x+z\|^2 + \|y+z\|^2 - \|x-z\|^2 - \|y-z\|^2\Big) = \langle x,z\rangle + \langle y,z\rangle.\end{aligned}$$

Let $n\in\mathbb{N}$. By induction on the number of summands in the first slot, we have

$$\langle nx,z\rangle = \Big\langle \sum_{k=1}^n x, z\Big\rangle = \sum_{k=1}^n \langle x,z\rangle = n\langle x,z\rangle.$$

Moreover, since $\langle 0,y\rangle = \frac14\Big(\|y\|^2 - \|y\|^2\Big) = 0$,

$$0 = \langle 0,y\rangle = \langle x-x,y\rangle = \langle x,y\rangle + \langle -x,y\rangle \qquad \Rightarrow\ \langle -x,y\rangle = -\langle x,y\rangle.$$

Consequently, $\langle mx,z\rangle = m\langle x,z\rangle$ for every $m\in\mathbb{Z}$. Let $m\in\mathbb{Z}$ and $n\in\mathbb{N}$. Then,

$$\Big\langle \frac{m}{n}x, z\Big\rangle = \frac{n}{n}m\Big\langle \frac1n x, z\Big\rangle = \frac{m}{n}\Big\langle \frac{n}{n}x, z\Big\rangle = \frac{m}{n}\langle x,z\rangle,$$

which implies $\langle qx, z\rangle = q\langle x, z\rangle$ for every $q \in \mathbb{Q}$.

Let $\lambda \in \mathbb{R}$ and let $(q_n)_{n\in\mathbb{N}}$ be a sequence of rational numbers converging to λ for $n \to \infty$. Since the triangle inequality $|\|x\| - \|y\|| \leq \|x - y\|$ implies that the norm is a continuous map, we have

$$\begin{aligned}\langle \lambda x, z\rangle &= \tfrac{1}{4}\|\lambda x + z\|^2 - \tfrac{1}{4}\|\lambda x - z\|^2 = \lim_{n\to\infty}\Big(\tfrac{1}{4}\|q_n x + z\|^2 - \tfrac{1}{4}\|q_n x - z\|^2\Big)\\ &= \lim_{n\to\infty}\langle q_n x, z\rangle = \lim_{n\to\infty} q_n\langle x, z\rangle = \lambda\langle x, z\rangle.\end{aligned}$$

Linearity in the second argument follows by symmetry.

- *Positive-definiteness.* For all $x \in V$, we have

$$\langle x, x\rangle = \tfrac{1}{4}\|x + x\|^2 - \tfrac{1}{4}\|x - x\|^2 = \tfrac{1}{4}\|2x\|^2 = \|x\|^2 \geq 0.$$

This also shows that $\|\cdot\|$ is induced by $\langle\cdot,\cdot\rangle$. Moreover, $\langle x, x\rangle = 0 \Leftrightarrow \|x\| = 0 \Leftrightarrow x = 0$.

(b) The metric $d(\cdot,\cdot)$ is induced by a norm $\|\cdot\|$ (in the sense that there exists a norm $\|\cdot\|$ such that $\forall x, y \in V: \ d(x, y) = \|x - y\|$)

$\Leftrightarrow$ the metric is *translation invariant* and *homogeneous*, i. e. $\forall v, x, y \in V \ \forall\lambda \in \mathbb{R}$:

$$\begin{aligned}d(x + v, y + v) &= d(x, y),\\ d(\lambda x, \lambda y) &= |\lambda| d(x, y).\end{aligned}$$

Solution: If the metric d is induced by the norm $\|\cdot\|$, then

$$\begin{aligned}d(x + v, y + v) &= \|(x + v) - (y + v)\| = \|x - y\| = d(x, y),\\ d(\lambda x, \lambda y) &= \|\lambda x - \lambda y\| = \|\lambda(x - y)\| = |\lambda|\|x - y\|.\end{aligned}$$

Conversely, we assume that the metric d is translation invariant and homogeneous and claim that

$$\|x\| := d(x, 0)$$

defines a norm which induces d. The function $\|\cdot\|$ is indeed a norm, because for all $x, y, z \in V$ and $\lambda \in \mathbb{R}$, we have

$$\begin{aligned}\|x\| = 0 \ &\Leftrightarrow \ d(x, 0) = 0 \ \Leftrightarrow \ x = 0,\\ \|\lambda x\| &= d(\lambda x, 0) = d(\lambda x, \lambda 0) = |\lambda| d(x, 0) = |\lambda|\|x\|,\\ \|x + y\| &= d(x + y, 0) \leq d(x + y, y) + d(y, 0) = d(x, 0) + d(y, 0) = \|x\| + \|y\|.\end{aligned}$$

Moreover, $\|\cdot\|$ induces the metric d since for all $x, y \in V$

$$\|x - y\| = d(x - y, 0) = d(x, y).$$

1.6. A classic

Let (X, d) be a non-empty complete metric space, let $\lambda \in [0, 1)$, and let $\Phi\colon X \to X$ be a mapping which satisfies for all $x, y \in X$ that $d(\Phi(x), \Phi(y)) \leq \lambda d(x, y)$. Show that there exists a unique $z \in X$ which satisfies $\Phi(z) = z$.

Solution: Let $x_0 \in X$ be arbitrary and let x_n for $n \in \mathbb{N}$ be given by $x_n = \Phi(x_{n-1})$. Note that for all $n \in \mathbb{N}$ it holds that

$$d(x_{n+1}, x_n) = d(\Phi(x_n), \Phi(x_{n-1})) \leq \lambda d(x_n, x_{n-1}).$$

Via induction, we obtain for all $n \in \mathbb{N}_0$ that $d(x_{n+1}, x_n) \leq \lambda^n d(x_1, x_0)$. The triangle inquality hence implies for all $n \in \mathbb{N}_0$, $k \in \mathbb{N}$ that

$$\begin{aligned} d(x_n, x_{n+k}) &\leq \sum_{l=0}^{k-1} d(x_{n+l+1}, x_{n+l}) \leq \sum_{l=0}^{k-1} \lambda^{n+l} d(x_1, x_0) \\ &= \lambda^n d(x_1, x_0) \sum_{l=0}^{k-1} \lambda^l \leq \frac{\lambda^n}{1-\lambda} d(x_1, x_0). \end{aligned}$$

This implies that $(x_n)_{n \in \mathbb{N}_0}$ is Cauchy in (X, d). Hence, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Note that the fact that Φ is Lipschitz continuous implies that

$$\Phi(z) = \Phi(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} \Phi(x_n) = \lim_{n\to\infty} x_{n+1} = z.$$

This finishes the existence part of the proof. The fact that z is the unique fixed point of Φ follows since for every $w \in X$ with $\Phi(w) = w$, we get:

$$d(z, w) = d(\Phi(z), \Phi(w)) \leq \lambda d(z, w),$$

which only holds if $d(z, w) = 0$ as $\lambda < 1$. Thus, $z = w$.