2.1. Statements of Baire

For a metric space (M, d) we shall prove the equivalence of

(i) Every residual set $\Omega \subseteq M$ is dense in M.

(ii) The interior of every meagre set $A \subseteq M$ is empty.

(iii) The empty set is the only subset of M which is open and meagre.

(iv) Countable intersections of dense open sets are dense.

Solution: "(i) $\Rightarrow$ (ii)": Let $A \subseteq M$ be a meagre set. Then, by definition, $A^{\complement}$ is residual. Moreover, by (i), $A^{\complement}$ is dense in M. Hence, $\emptyset = (M \setminus A^{\complement})^\circ = A^\circ$.

"(ii) $\Rightarrow$ (iii)": Let $A \subseteq M$ be open and meagre. Then $A = A^\circ$ and, by (ii), $A^\circ = \emptyset$.

"(iii) $\Rightarrow$ (iv)": Let $A = \bigcap_{n\in\mathbb{N}} A_n$ be a countable intersection of dense open sets $A_n \subseteq M$, $n \in \mathbb{N}$. Fix an arbitrary $n \in \mathbb{N}$. Since A_n is dense, $(A_n^{\complement})^\circ = \emptyset$. Since A_n is open, $A_n^{\complement}$ is closed. Thus, $(\overline{A_n^{\complement}})^\circ = (A_n^{\complement})^\circ = \emptyset$, which means that $A_n^{\complement}$ is nowhere dense. Thus, $A^{\complement} = \bigcup_{k\in\mathbb{N}} A_k^{\complement}$ is meagre. $(A^{\complement})^\circ$ is open and meagre, hence empty by (iii). This implies that A is dense in M.

"(iv) $\Rightarrow$ (i)": Let $\Omega \subseteq M$ be a residual set. Since $A = \Omega^{\complement}$ is meagre, $A = \bigcup_{n\in\mathbb{N}} A_n$ for nowhere dense sets A_n, $n \in \mathbb{N}$. Then it holds for every $n \in \mathbb{N}$ that $\emptyset = (\overline{A_n})^\circ = (M \setminus (\overline{A_n})^{\complement})^\circ$ which implies that $(\overline{A_n})^{\complement}$ is dense in M for all $n \in \mathbb{N}$. Moreover, all the sets $(\overline{A_n})^{\complement}$, $n \in \mathbb{N}$, are open since $\overline{A_n}$ is closed for every $n \in \mathbb{N}$. Then, (iv) implies density of $\bigcap_{n\in\mathbb{N}} (\overline{A_n})^{\complement}$, which in turn implies the density of Ω by the following chain of (in-)equalities:

$$\Omega = A^{\complement} = \bigcap_{n\in\mathbb{N}} A_n^{\complement} \supseteq \bigcap_{n\in\mathbb{N}} (\overline{A_n})^{\complement}.$$

2.2. Algebraic (Hamel) bases for Banach spaces

Let X be a vector space. An *algebraic basis* for X is a subset $E \subseteq X$ such that every $x \in X$ is uniquely given as *finite* linear combination of elements in E.

(a) Show that, if $(X, \|\cdot\|)$ is a Banach space, then any algebraic basis for X is either finite or uncountable.

Solution: Assume by contradiction that X has a countably infinite algebraic basis $\{e_1, e_2, \ldots\}$. For $n \in \mathbb{N}$ we define the linear subspaces $A_n = \operatorname{span}\{e_1, \ldots, e_n\} \subseteq X$.

For every $n \in \mathbb{N}$, we find that, being a finite dimensional subspace, A_n is closed (why?). Suppose that there exists $n \in \mathbb{N}$ such that A_n has non-empty interior. Then there exist $x \in A_n$ and $\varepsilon \in (0, \infty)$ such that $B_\varepsilon(x) \subseteq A_n$, where $B_\varepsilon(x)$ denotes the open ε-ball with center x. Since A_n is a linear subspace, we may subtract $x \in A_n$ from the elements in $B_\varepsilon(x)$ to obtain $B_\varepsilon(0) \subseteq A_n$. For the same reason,

$$A_n \supseteq \{\lambda y \mid \lambda > 0,\ y \in B_\varepsilon(x)\} = X.$$

This implies $\dim X \leq n$ which contradicts our assumption that the algebraic basis of X is infinite. Thus, for every $n \in \mathbb{N}$, the set A_n must have empty interior and, being also closed, needs to be nowhere dense. By assumption,

$$X = \bigcup_{n\in\mathbb{N}} A_n,$$

which implies that X is meager. Since X is complete, this contradicts Baire's Theorem.

(b) Let $\mathcal{P}$ be the vector space of all real-valued polynomials over $\mathbb{R}$, i.e.,

$$\mathcal{P} = \left\{ p\colon \mathbb{R} \to \mathbb{R} \mid \exists n \in \mathbb{N}_0, a_0, a_1, \ldots, a_n \in \mathbb{R}\colon \forall t \in \mathbb{R}\colon p(t) = \sum_{k=0}^{n} a_k t^k \right\}.$$

Show that there is no norm $\|\cdot\|\colon \mathcal{P} \to [0,\infty)$ on $\mathcal{P}$ turning $(\mathcal{P}, \|\cdot\|)$ into a Banach space.

Solution: With the monomials $\{\mathbb{R} \ni x \mapsto x^n \in \mathbb{R} \mid n \in \mathbb{N}\} \subseteq \mathcal{P}$ constituting a countably infinite algebraic basis of $\mathcal{P}$, the previous part implies that $(\mathcal{P}, \|\cdot\|)$ cannot be a Banach space, no matter what the norm $\|\cdot\|$ is.

2.3. A real analysis application

Let $f \in C^0([0,\infty))$ be a continuous function satisfying

$$\forall t \in [0,\infty): \lim_{n\to\infty} f(nt) = 0.$$

Prove that $\lim_{t\to\infty} f(t) = 0$.

Solution: Given $f \in C^0([0,\infty))$ satisfying $\forall t \in [0,\infty): \lim_{n\to\infty} f(nt) = 0$ we define the functions $f_n\colon [0,\infty) \to \mathbb{R}$, $n \in \mathbb{N}$, via $f_n(t) = |f(nt)|$ for all $t \in [0,\infty)$, $n \in \mathbb{N}$. Let $\varepsilon \in (0,\infty)$ and define for every $N \in \mathbb{N}$ the set

$$A_N := \bigcap_{n=N}^{\infty} \{t \in [0,\infty) \mid f_n(t) \leq \varepsilon\}.$$

Since for every $n \in \mathbb{N}$ the function f_n is continuous, we have that the pre-image $f_n^{-1}([0,\varepsilon]) = \{t \in [0,\infty) \mid f_n(t) \leq \varepsilon\}$ is closed for all $n \in \mathbb{N}$. Thus, the set A_N is closed as intersection of closed sets. By assumption,

$$\forall t \in [0,\infty) \quad \exists N_t \in \mathbb{N} \quad \forall n \geq N_t: \quad f_n(t) \leq \varepsilon$$

(i.e., $\forall t \in [0,\infty)\, \exists N_t \in \mathbb{N}\colon t \in A_{N_t}$) which implies

$$[0,\infty) = \bigcup_{N=1}^{\infty} A_N.$$

Baire's Theorem, applied to the complete metric space $([0,\infty), |\cdot|)$, implies that there exists $N_0 \in \mathbb{N}$ such that A_{N_0} has non-empty interior, i.e., there exist $0 \le a < b$ such that $(a,b) \subseteq A_{N_0}$. This implies

$$\begin{aligned} & \forall n \ge N_0 \quad \forall t \in (a,b): \quad f_n(t) \le \varepsilon \\ \Leftrightarrow \quad & \forall n \ge N_0 \quad \forall t \in (na,nb): \quad |f(t)| \le \varepsilon. \end{aligned}$$

If $n > \frac{a}{b-a}$, then $(n+1)a < nb$. For the intervals $J_{a,b}(n) := (na,nb)$ this means that $J_{a,b}(n) \cap J_{a,b}(n+1) \neq \emptyset$. Letting $N_1 > \max\{N_0, \frac{a}{b-a}\}$, we therefore obtain

$$\forall t > N_1 a: \qquad |f(t)| \le \varepsilon.$$

This proves $\lim\limits_{t\to\infty} f(t) = 0$ since $\varepsilon \in (0,\infty)$ was arbitrary.

2.4. Singularity condensation

Let $(X, \|\cdot\|_X)$ be a Banach space and let $(Y_1, \|\cdot\|_{Y_1}), (Y_2, \|\cdot\|_{Y_2}), \ldots$ be normed spaces. For every $n \in \mathbb{N}$, let $G_n \subseteq L(X, Y_n)$ be an unbounded set of linear continuous mappings from X to Y_n. Prove that there exists $x \in X$ satisfying for all $n \in \mathbb{N}$ that $\sup_{T\in G_n} \|Tx\|_{Y_n} = \infty$.

Solution: Assume for a contradiction that for every $x \in X$ there exists $n \in \mathbb{N}$ such that $\sup_{T\in G_n} \|Tx\|_{Y_n} < \infty$. In other words, we can write $X = \bigcup_{n\in\mathbb{N}} A_n$ with A_n defined for every $n \in \mathbb{N}$ via

$$A_n = \left\{ x \in X\colon \sup_{T\in G_n} \|Tx\|_{Y_n} < \infty \right\}.$$

Note that X would be meagre if all the sets A_n, $n \in \mathbb{N}$, were meagre. Since X is not meagre by Baire's Theorem, we thus conclude that there exists $N \in \mathbb{N}$ so that A_N is not meagre. Observe that A_N may be represented as $A_N = \bigcup_{k\in\mathbb{N}} B_k$ with B_k defined for every $k \in \mathbb{N}$ by

$$B_k = \left\{ x \in X\colon \sup_{T\in G_N} \|Tx\|_{Y_N} \le k \right\}.$$

The assumption that G_N is a set of linear continuous mappings from X to Y_N implies that B_k is closed for every $k \in \mathbb{N}$. Together with the fact that A_N is not meagre, we deduce the existence of $K \in \mathbb{N}$ such that $B_K^\circ \neq \emptyset$. That is, there exist $x \in X$, $\varepsilon \in (0,\infty)$ so that $\{y \in X : \|y-x\|_X < \varepsilon\} \subseteq B_k$. This, on the other hand, implies for every $y \in X$ with $\|y\|_X \le 1$ that

$$\begin{aligned}\|Ty\|_{Y_N} &= \frac{2}{\varepsilon}\left\|T\Big(x+\frac{\varepsilon}{2}y-x\Big)\right\|_{Y_N}\\ &\le \frac{2}{\varepsilon}\left(\left\|T\Big(x+\frac{\varepsilon}{2}y\Big)\right\|_{Y_N}+\|Tx\|_{Y_N}\right) \le \frac{4k}{\varepsilon}, \quad \text{for all } T \in G_N.\end{aligned}$$

This, on the other hand, contradicts the assumption that G_N is unbounded.

2.5. Discrete L^p-spaces and inclusions

Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be a sequence. Define, for every $p \in [1,\infty]$,

$$\|(x_n)_{n\in\mathbb{N}}\|_{\ell^p} = \begin{cases}\left(\sum_{n\in\mathbb{N}} |x_n|^p\right)^{1/p} & \text{if } p<\infty,\\ \sup_{n\in\mathbb{N}} |x_n| & \text{if } p=\infty,\end{cases}$$

and let $\ell^p = \{(x_n)_{n\in\mathbb{N}} \mid \|(x_n)_{n\in\mathbb{N}}\|_{\ell^p} < \infty\}$.

(a) Show for every $p \in [1,\infty]$ that $(\ell^p, \|\cdot\|_{\ell^p})$ is a Banach space.

Solution: Let $(x_n)_{n\in\mathbb{N}} \subseteq \ell^p$ with $x_n = (x_{n,k})_{k\in\mathbb{N}}$, $n \in \mathbb{N}$, be a Cauchy sequence. This implies that for every $k \in \mathbb{N}$, the sequence $(x_{n,k})_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is Cauchy in $\mathbb{R}$. Hence, there exists a sequence $(a_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}$ satisfying for every $k \in \mathbb{N}$ that $\limsup_{n\to\infty} |x_{n,k}-a_k| = 0$. It remains to show that $(a_k)_{k\in\mathbb{N}} \in \ell^p$ and that $(x_n)_{n\in\mathbb{N}} \subseteq \ell^p$ converges to $(a_k)_{k\in\mathbb{N}}$. We argue in the following only for $p \in [1,\infty)$ and leave the case $p=\infty$ as an exercise. The fact that Cauchy sequences are bounded and the fact that $(x_{n,k})_{n\in\mathbb{N}}$ converges to a_k for every $k \in \mathbb{N}$ imply for every $N \in \mathbb{N}$ that

$$\sum_{k=1}^N |a_k|^p = \lim_{n\to\infty} \sum_{k=1}^N |x_{n,k}|^p \le \sup_{n\in\mathbb{N}} \|x_n\|_{\ell^p}^p < \infty.$$

Letting $N \to \infty$ establishes $(a_k)_{k\in\mathbb{N}} \in \ell^p$. Moreover, for every $N \in \mathbb{N}$, we have:

$$\sum_{k=1}^N |a_k - x_{n,k}|^p = \lim_{m\to\infty} \sum_{k=1}^N |x_{m,k}-x_{n,k}|^p \le \sup_{m\ge n} \|x_m - x_n\|_{\ell^p}^p.$$

Hence, for all $n \in \mathbb{N}$, it holds that $\|a - x_n\|_{\ell^p} \le \sup_{m\ge n} \|x_m - x_n\|_{\ell^p}$. Since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, the right-hand side tends to 0 as $n \to \infty$.

Let now $1 \le p < q \le \infty$. Prove that:

(b) $\ell^p \subsetneq \ell^q$ and $\|(x_n)_{n\in\mathbb{N}}\|_{\ell^q} \leq \|(x_n)_{n\in\mathbb{N}}\|_{\ell^p}$ for every $(x_n)_{n\in\mathbb{N}} \in \ell^p$.

Solution: It suffices to prove the inequality $\|x\|_{\ell^q} \leq \|x\|_{\ell^p}$ for all $x \in \ell^p$ which implies the inclusion $\ell^p \subseteq \ell^q$ by definition of the spaces. Since $(n^{-\frac{1}{p}})_{n\in\mathbb{N}} \in \ell^q \setminus \ell^p$, the inclusion is strict.

Scaling. Since $\|x\|_{\ell^q} \leq \|x\|_{\ell^p}$ if and only if $\|\lambda x\|_{\ell^q} \leq \|\lambda x\|_{\ell^p}$ for some $\lambda > 0$, it suffices to prove $\|x\|_{\ell^q} \leq 1$ for all $x = (x_n)_{n\in\mathbb{N}} \in \ell^p$ with $\|x\|_{\ell^p} = 1$.

Case $q = \infty$. For all $n \in \mathbb{N}$ we have

$$|x_n| = \Big(|x_n|^p\Big)^{\frac{1}{p}} \leq \Big(\sum_{k=1}^{\infty}|x_k|^p\Big)^{\frac{1}{p}} = \|x\|_{\ell^p} = 1.$$

Therefore, $\|x\|_{\ell^\infty} = \sup_{n\in\mathbb{N}}|x_n| \leq 1$.

Case $q < \infty$. The assumption $\|x\|_{\ell^p} = 1$ implies $|x_n| \leq 1$ for all $n \in \mathbb{N}$. Since $1 \leq p < q$, we have $|x_n|^q \leq |x_n|^p$ for all $n \in \mathbb{N}$. This implies the inequality

$$\|x\|_{\ell^q} = \Big(\sum_{n\in\mathbb{N}}|x_n|^q\Big)^{\frac{1}{q}} \leq \Big(\sum_{n\in\mathbb{N}}|x_n|^p\Big)^{\frac{1}{q}} = \Big(\|x\|_{\ell^p}^p\Big)^{\frac{1}{q}} = 1^{\frac{p}{q}} = 1.$$

(c) ℓ^p is meager in ℓ^q.

Solution: Define for every $n \in \mathbb{N}$ the set A_n as $A_n = \{x \in \ell^q \mid \|x\|_{\ell^p} \leq n\}$. It is clear that $\ell^q = \bigcup_{n\in\mathbb{N}} A_n$. It is our goal to show for every $n \in \mathbb{N}$ that A_n is closed and has empty interior.

Let $n \in \mathbb{N}$ be arbitrary but fixed. In order to show that A_n is closed in $(\ell^q, \|\cdot\|_{\ell^q})$, we will prove that the limit of every ℓ^q-convergent sequence with elements in A_n is also in A_n. Let $(a^{(k)})_{k\in\mathbb{N}}$ be a sequence of elements $a^{(k)} = (a_j^{(k)})_{j\in\mathbb{N}} \in A_n$, $k \in \mathbb{N}$. Suppose $b = (b_j)_{j\in\mathbb{N}} \in \ell^q$ satisfies $\lim_{k\to\infty}\|a^{(k)} - b\|_{\ell^q} = 0$. Then, for every $j \in \mathbb{N}$,

$$|a_j^{(k)} - b_j| \leq \Big(\sum_{i\in\mathbb{N}}|a_i^{(k)} - b_i|^q\Big)^{\frac{1}{q}} = \|a^{(k)} - b\|_{\ell^q} \xrightarrow{k\to\infty} 0.$$

Let $N \in \mathbb{N}$ be arbitrary. By continuity of $|\cdot|^p \colon \mathbb{R} \to \mathbb{R}$, we have

$$\sum_{j=1}^{N}|b_j|^p = \lim_{k\to\infty}\sum_{j=1}^{N}|a_j^{(k)}|^p \leq \limsup_{k\to\infty}\|a^{(k)}\|_{\ell^p}^p \leq n^p$$

since the number of summands is finite. In the limit $N \to \infty$, we see $\|b\|_{\ell^p}^p \leq n^p$, which implies $b \in A_n$. Therefore, A_n is closed in $(\ell^q, \|\cdot\|_{\ell^q})$.

Next we show that $A_n^\circ = \emptyset$. Towards a contradiction, suppose A_n has non-empty interior in the ℓ^q- topology. Then there exist $a = (a_m)_{m\in\mathbb{N}} \in A_n$ and $\varepsilon > 0$ such that

$$B := \{x \in \ell^q \mid \|a - x\|_{\ell^q} < \varepsilon\} \subseteq A_n.$$

Consider $b = (b_m)_{m\in\mathbb{N}} \in \ell^q$ given by $b_m = m^{-\frac{1}{p}}$. Indeed, $\sum_{m=1}^{\infty} m^{-\frac{q}{p}} < \infty$ since $p < q$. We define $z = (z_m)_{m\in\mathbb{N}}$ by

$$z_m = a_m + \frac{\varepsilon b_m}{2\|b\|_{\ell^q}}.$$

Then $\|a - z\|_{\ell^q} = \frac{\varepsilon}{2}$ and $z \in B$. However, $b \notin \ell^p$ and $a \in \ell^p$ imply $z \notin \ell^p \supseteq A_n$ which contradicts $B \subseteq A_n$. Therefore, A_n has empty interior in $(\ell^q, \|\cdot\|_{\ell^q})$. Being closed with empty interior, A_n is nowhere dense in $(\ell^q, \|\cdot\|_{\ell^q})$.

Since $\ell^p = \bigcup_{n\in\mathbb{N}} A_n$ we may conclude that ℓ^p is meagre in ℓ^q.

(d) $\bigcup_{1\le r<q} \ell^r \subsetneq \ell^q$.

Solution: Since $\ell^{p_1} \subseteq \ell^{p_2}$ for $p_1 < p_2$ by (b) we have

$$\bigcup_{p\in[1,q[} \ell^p = \bigcup_{p\in[1,q[\cap\mathbb{Q}} \ell^p.$$

By (c), the right hand side is a countable union of meagre subsets of $(\ell^q, \|\cdot\|_{\ell^q})$ and therefore meagre itself (see lecture notes, Beispiel 1.3.2.iii). Being complete, ℓ^q is not meagre in $(\ell^q, \|\cdot\|_{\ell^q})$. Therefore, we may conclude strict inclusion

$$\bigcup_{p\in[1,q[\cap\mathbb{Q}} \ell^p \subsetneq \ell^q.$$

2.6. A reformulation of completeness for Banach spaces

Let $(X, \|\cdot\|)$ be a normed vector space. Prove that the following statements are equivalent.

(a) $(X, \|\cdot\|)$ is a Banach space.

(b) For every sequence $(x_n)_{n\in\mathbb{N}}$ in X with $\displaystyle\sum_{k=1}^{\infty} \|x_n\| < \infty$ the limit $\displaystyle\lim_{N\to\infty} \sum_{n=1}^{N} x_n$ exists.

Solution: If $(X, \|\cdot\|)$ is a Banach space, and $(x_k)_{k\in\mathbb{N}}$ any sequence in X with $\sum_{k=1}^{\infty}\|x_k\| < \infty$, then $(s_n)_{n\in\mathbb{N}}$ given by $s_n = \sum_{k=1}^{n} x_k$ is a Cauchy sequence (and hence convergent) since by assumption, for every $\varepsilon \in (0, \infty)$ there exists $N_\varepsilon \in \mathbb{N}$ such that for every $m \geq n \geq N_\varepsilon$,

$$\|s_m - s_n\| \leq \sum_{k=n+1}^{m} \|x_k\| \leq \sum_{k=N_\varepsilon+1}^{\infty} \|x_k\| < \varepsilon.$$

Conversely, we assume for every sequence $(x_k)_{k\in\mathbb{N}}$ in X that $\sum_{k=1}^{\infty}\|x_k\| < \infty$ implies convergence of $s_n = \sum_{k=1}^{n} x_k$ in X for $n \to \infty$. Let $(y_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X. Then,

$$\forall k \in \mathbb{N} \quad \exists N_k \in \mathbb{N} \quad \forall n, m \geq N_k : \quad \|y_n - y_m\| \leq 2^{-k}.$$

Without loss of generality, we can assume $N_{k+1} > N_k$. Let $x_k := y_{N_{k+1}} - y_{N_k}$. Then,

$$\sum_{k=1}^{\infty}\|x_k\| = \sum_{k=1}^{\infty}\|y_{N_{k+1}} - y_{N_k}\| \leq \sum_{k=1}^{\infty} 2^{-k} < \infty,$$

which by assumption implies that the sequence $(s_n)_{n\in\mathbb{N}} \subseteq X$, given by

$$s_n = \sum_{k=1}^{n} x_k = \sum_{k=1}^{n}(y_{N_{k+1}} - y_{N_k}) = y_{N_{n+1}} - y_{N_1}, \quad \text{for all } n \in \mathbb{N},\text{Í}$$

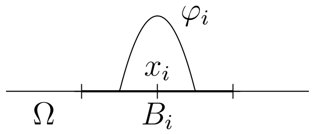
converges in X for $n \to \infty$. Hence, $(y_{N_n})_{n\in\mathbb{N}}$ is a convergent subsequence of $(y_n)_{n\in\mathbb{N}}$. Since $(y_n)_{n\in\mathbb{N}}$ is Cauchy, it converges to the same limit in X. Thus, $(X, \|\cdot\|)$ is complete.

2.7. Infinite-dimensional vector spaces and separability

(a) Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be an open set. Show that $L^p(\Omega)$ is an infinite-dimensional vector space for all $1 \leq p \leq \infty$.

Solution: Suppose by contradiction, $L^p(\Omega)$ has finite dimension $d \in \mathbb{N}$. Since $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is open there exist $d+1$ disjoint balls $B_i := B_{r_i}(x_i) \subseteq \Omega$ for $i = 1, \ldots, d+1$. For every $i \in \{1, 2, \ldots, d+1\}$ let $\varphi_i \colon \Omega \to \mathbb{R}$ be given by

$$\varphi_i(x) = \max\left\{0, 1 - \frac{4|x - x_i|^2}{r_i^2}\right\}.$$



Then, $\varphi_1, \dots, \varphi_{d+1} \in C_c(\Omega) \subseteq L^p(\Omega)$ with disjoint supports. Moreover, since the subset $\{\varphi_1, \dots, \varphi_{d+1}\}$ contains more than d elements, it must be linearly dependent. Let $\lambda_1, \dots, \lambda_{d+1} \in \mathbb{R}$ be not all equal to 0 such that

$$\sum_{i=1}^{d+1} \lambda_i \varphi_i = 0.$$

However, if we multiply by φ_j for any $j \in \{1, \dots, d+1\}$ and integrate over Ω,

$$0 = \int_\Omega \sum_{i=1}^{d+1} \lambda_i \varphi_i \varphi_j \, d\mu = \int_\Omega \lambda_i \varphi_j^2 \, d\mu = \lambda_j \int_\Omega \varphi_j^2 \, d\mu \qquad \Rightarrow \lambda_j = 0.$$

(b) Let $(X, \mathcal{A}, \mu)$ be a measure space. Recall that if X is separable and the measure μ is finite (or, more generally, σ-finite) and if $1 \le p < \infty$, then the space $L^p(X, \mathcal{A}, \mu)$ is separable. Roughly speaking, in the simple case when $X = (0,1)$, $\mathcal{A}$ = Borel-σ-algebra and $\mu = \mathcal{L}^1$, this relies on the fact that any element in those spaces can be arbitrarily well approximated by a function of the form

$$f = \sum_{i=1}^{k} q_i \, \chi_{B_i} \quad \text{for } k \in \mathbb{N},\ B_i := B_{r_i}(x_i),\ q_i \in \mathbb{Q},\ x_i \in \mathbb{Q} \cap (0,1),\ 0 < r_i \in \mathbb{Q}.$$

Show that instead $\Big(L^\infty((0,1)), \|\cdot\|_{L^\infty((0,1))}\Big)$ is *not* separable, i.e., it does not contain a countable dense subset.

Solution: We define $I_n := (\frac{1}{n+1}, \frac{1}{n}) \subseteq (0,1)$ for $n \in \mathbb{N}$ and consider the characteristic function χ_{I_n} of I_n, i. e.

$$\chi_{I_n}(x) := \begin{cases} 1, & \text{if } x \in I_n, \\ 0, & \text{if } x \in (0,1) \setminus I_n. \end{cases}$$

Given any subset $\emptyset \neq M \subseteq \mathbb{N}$ we define the function $f_M \in L^\infty((0,1))$ by

$$f_M(x) := \sum_{n \in M} \chi_{I_n}(x)$$

Since the intervals I_n, $n \in \mathbb{N}$, are pairwise disjoint, open and non-empty, we have $\|f_M\|_{L^\infty} = 1$ for every $\emptyset \neq M \subseteq \mathbb{N}$. For the same reason,

$$\|f_M - f_{M'}\|_{L^\infty} = 1.$$

if $M \neq M'$. Therefore, the balls $B_M = \{g \in L^\infty((0,1)) \mid \|g - f_M\|_{L^\infty} < \frac{1}{3}\}$ are pairwise disjoint. If $S \subseteq L^\infty((0,1))$ is any dense subset, then $S \cap B_M \neq \emptyset$ for every $\emptyset \neq M \subseteq \mathbb{N}$. Thus, there is a surjective map $S \to \{B_M \mid \emptyset \neq M \subseteq \mathbb{N}\}$. Since there are uncountably many different subsets of $\mathbb{N}$, the set S must be uncountable as well. Therefore, $L^\infty((0,1))$ does not admit a countable dense subset.