3.1. The space of bounded linear operators

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed $\mathbb{K}$-vector spaces with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $L(X,Y)$ be the space of bounded $\mathbb{K}$-linear operators $T\colon X \to Y$, equipped with the norm $\|\cdot\|_{L(X,Y)}\colon L(X,Y) \to [0,\infty)$, defined by

$$\|T\|_{L(X,Y)} = \sup_{x\neq 0} \frac{\|Tx\|_Y}{\|x\|_X} \quad \text{for all } T \in L(X,Y).$$

(a) Prove that

$$\|T\|_{L(X,Y)} = \sup_{\|x\|_X \leq 1} \|Tx\|_Y = \sup_{\|x\|_X = 1} \|Tx\|_Y \quad \text{for all } T \in L(X,Y).$$

Solution: Linearity of T and the fact that $X \setminus \{0\} = \{\lambda x \colon \lambda \in \mathbb{K} \setminus \{0\}, \|x\|_X = 1\}$ imply

$$\begin{aligned}\sup_{x\neq 0} \frac{\|Tx\|_Y}{\|x\|_X} &= \sup_{\|x\|_X = 1, \lambda \in \mathbb{K}\setminus\{0\}} \frac{\|T(\lambda x)\|_Y}{\|\lambda x\|_X} = \sup_{\|x\|_X = 1, \lambda \in \mathbb{K}\setminus\{0\}} \frac{\|Tx\|_Y}{\|x\|_X} \\ &= \sup_{\|x\|_X = 1} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{\|x\|_X = 1} \|Tx\|_Y.\end{aligned}$$

Moreover, due to $\{x \in X \colon \|x\|_X \leq 1\} = \{\lambda x \colon |\lambda| \leq 1, \|x\|_X = 1\}$ we obtain

$$\sup_{\|x\|_X \leq 1} \|Tx\|_Y = \sup_{|\lambda|\leq 1, \|x\|_X = 1} \|T(\lambda x)\|_Y = \sup_{|\lambda|\leq 1, \|x\|_X = 1} \lambda \|Tx\|_Y = \sup_{\|x\|_X = 1} \|Tx\|_Y.$$

(b) Prove that $\|\cdot\|_{L(X,Y)}$ is indeed a norm on $L(X,Y)$.

Solution: Clearly, $\|\cdot\|_{L(X,Y)}\colon L(X,Y) \to [0,\infty)$ is well-defined. Moreover, $\|T\|_{L(X,Y)} = 0$ for $T \in L(X,Y)$ implies that $\|Tx\|_Y \leq 0\|x\|_X = 0$ for all $x \in X$, i.e., $Tx = 0 \in Y$ for all $x \in X$, which just means $T = 0 \in L(X,Y)$. Next, note that, by (a), we have for all $\lambda \in \mathbb{K}$, $T \in L(X,Y)$ that

$$\begin{aligned}\|\lambda T\|_{L(X,Y)} &= \sup_{\|x\|_X \leq 1} \|(\lambda T)x\|_Y = \sup_{\|x\|_X \leq 1} \|\lambda Tx\|_Y = \sup_{\|x\|_X \leq 1} |\lambda| \|Tx\|_Y \\ &= |\lambda| \|T\|_{L(X,Y)}.\end{aligned}$$

In addition, we obtain for $S, T \in L(X,Y)$:

$$\begin{aligned}\|S+T\|_{L(X,Y)} &= \sup_{\|x\|_X \leq 1} \|(S+T)x\|_Y = \sup_{\|x\|_X \leq 1} \|Sx + Tx\|_Y \\ &\leq \sup_{\|x\|_X \leq 1} (\|Sx\|_Y + \|Tx\|_Y) \leq \sup_{\|x\|_X \leq 1} \|Sx\|_Y + \sup_{\|x\|_X \leq 1} \|Tx\|_Y \\ &= \|S\|_{L(X,Y)} + \|T\|_{L(X,Y)}.\end{aligned}$$

(c) Prove that $(L(X,Y), \|\cdot\|_{L(X,Y)})$ is a $\mathbb{K}$-Banach space if and only if $(Y, \|\cdot\|_Y)$ is a $\mathbb{K}$-Banach space or $X = \{0\}$.

Solution: Let us assume that Y is a $\mathbb{K}$-Banach space. Let $(T_k)_{k\in\mathbb{N}} \subseteq L(X,Y)$ be an arbitrary Cauchy sequence in $(L(X,Y), \|\cdot\|_{L(X,Y)})$. This implies for every $x \in X$ that $(T_k x)_{k\in\mathbb{N}} \subseteq Y$ is Cauchy in $(Y, \|\cdot\|_Y)$. By the completeness of $(Y, \|\cdot\|_Y)$, there exists a map $T_\infty \colon X \to Y$ (which a priori does not need to be linear) satisfying for every $x \in X$ that $\limsup_{k\to\infty} \|T_k x - T_\infty(x)\|_Y = 0$. The linearity of the mappings $(T_k)_{k\in\mathbb{N}}$, though, ensures that T_∞ is also a linear map. The fact that Cauchy sequences are bounded implies that

$$\|T_\infty x\|_Y \le \sup_{n\in\mathbb{N}} \|T_n\|_{L(X,Y)} \|x\|_X \quad \text{for all } n \in \mathbb{N},$$

i.e., that T_∞ is a bounded linear map. Thus, $T_\infty \in L(X,Y)$. Moreover, due to the sequence $(T_k)_{k\in\mathbb{N}}$ being Cauchy in $(L(X,Y), \|\cdot\|_{L(X,Y)})$, there exists $N \colon (0,\infty) \to \mathbb{N}$ such that for every $\varepsilon \in (0,\infty)$ it holds that $\sup_{k,m\ge N_\varepsilon} \|T_k - T_m\|_{L(X,Y)} \le \varepsilon$. By T_∞ being the pointwise (sometimes also called *strong*) limit of the sequence $(T_k)_{k\in\mathbb{N}}$, there exists $M \colon (0,\infty) \times X \to \mathbb{N}$ such that for all $\varepsilon \in (0,\infty)$, $x \in X$ it holds that $\sup_{m \ge M_{\varepsilon,x}} \|T_m x - T_\infty x\|_Y \le \varepsilon$. Using these, we get for every $\varepsilon \in (0,\infty)$ the following estimate:

$$\begin{aligned}
&\sup_{k\ge N_\varepsilon} \sup_{\|x\|_X \le 1} \|T_k x - T_\infty x\|_Y \\
&\le \sup_{k\ge N_\varepsilon} \sup_{\|x\|_X \le 1} \Big[\|T_k x - T_{\max\{N_\varepsilon, M_{\varepsilon,x}\}} x\|_Y + \|T_{\max\{N_\varepsilon, M_{\varepsilon,x}\}} x - T_\infty x\|_Y \Big] \\
&\le \sup_{k\ge N_\varepsilon} \Big[\|T_k - T_{\max\{N_\varepsilon, M_{\varepsilon,x}\}}\|_{L(X,Y)} + \varepsilon \Big] \le 2\varepsilon.
\end{aligned}$$

Thus, we obtain $T_k \to T_\infty$ in $(L(X,Y), \|\cdot\|_{L(X,Y)})$ as $k \to \infty$, which completes the proof that $(L(X,Y), \|\cdot\|_{L(X,Y)})$ is complete.

For the converse, let us assume that $(L(X,Y), \|\cdot\|_{L(X,Y)})$ is a Banach space and that $X \neq \{0\}$. Let $(y_n)_{n\in\mathbb{N}} \subseteq Y$ by a Cauchy sequence in $(Y, \|\cdot\|_Y)$. Moreover, let $x_0 \in X \setminus \{0\}$ with $\|x_0\|_X = 1$ be fixed. The theorem of Hahn–Banach implies that there exists a continuous linear functional $\varphi \in X^* = L(X,\mathbb{K})$ satisfying $\|\varphi\|_{L(X,\mathbb{K})} = 1$ and $\varphi(x_0) = \|x_0\|_X = 1$. Define now for every $n \in \mathbb{N}$ the continuous linear mapping $T_n \colon X \to Y$ by setting $T_n x = \varphi(x) y_n$ for every $x \in X$. Note that $(T_n)_{n\in\mathbb{N}} \subseteq L(X,Y)$ is Cauchy in $(L(X,Y), \|\cdot\|_{L(X,Y)})$ due to $(y_n)_{n\in\mathbb{N}}$ being Cauchy in $(Y, \|\cdot\|_Y)$. Hence, there exists $T_\infty \in L(X,Y)$ such that $T_n \to T_\infty$ in $(L(X,Y), \|\cdot\|_{L(X,Y)})$ as $n \to \infty$. This implies in particular, that $y_n = T_n x_0 \to T_\infty x_0 =: y_\infty$ in $(Y, \|\cdot\|_Y)$.

(d) Prove that the dual space $L(X,\mathbb{K})$ of X is complete.

Solution: This follows immediately from (c) and the completeness of $(\mathbb{K}, |\cdot|)$.

3.2. Lipschitz functions

Let $X = \mathrm{Lip}([0,1],\mathbb{R})$ be the vector space of Lischitz continuous functions from $[0,1]$ to $\mathbb{R}$ and let $Y = C^1([0,1],\mathbb{R})$ be the vector space of continuously differentiable functions from $[0,1]$ to $\mathbb{R}$. Define the functions $\|\cdot\|_{\mathrm{Lip}} \colon X \to [0,\infty)$ and $\|\cdot\|_{\mathrm{C}^1} \colon Y \to [0,\infty)$ by

$$\|x\|_{\mathrm{Lip}} = \sup_{s\in[0,1]} |x(s)| + \sup_{\substack{s,t\in[0,1]\\ s\neq t}} \left|\frac{x(s)-x(t)}{s-t}\right| \quad \text{for all } x \in X,$$

$$\|y\|_{\mathrm{C}^1} = \sup_{s\in[0,1]} |x(s)| + \sup_{s\in[0,1]} |x'(s)| \quad \text{for all } y \in Y.$$

(a) Prove that $\|\cdot\|_{\mathrm{Lip}}$ is a norm on X.

Solution: This is left to the interested reader.

(b) Show that $(X, \|\cdot\|_{\mathrm{Lip}})$ is a Banach space.

Solution: Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a Cauchy sequence. This entails in particular that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(C([0,1],\mathbb{R}), \|\cdot\|_{C([0,1],\mathbb{R})})$. Hence, there exists $x_\infty \in C([0,1],\mathbb{R})$ such that $(x_n)_{n\in\mathbb{N}}$ converges uniformly to x_∞. Moreover, boundedness of Cauchy sequences implies that $\sup_{n\in\mathbb{N}} \sup_{s,t\in[0,1],s\neq t} \left|\frac{x_n(s)-x_n(t)}{s-t}\right| < \infty$, i.e., there exists $L \in \mathbb{R}$ such that we have for all $n \in \mathbb{N}$, $s,t \in [0,1]$ that $|x_n(s) - x_n(t)| \leq L|s-t|$. This implies that x_∞ is Lipschitz with Lipschitz constant bounded by L. Finally, it remains to show that the convergence is also in $(X, \|\cdot\|_{\mathrm{Lip}})$. For this, note that – due to the Cauchy property – there exists $N \colon (0,\infty) \to \mathbb{N}$ so that for all $\varepsilon \in (0,\infty)$ and all $m,n \geq N_\varepsilon$ it holds that $\sup_{s,t\in[0,1],s\neq t} \left|\frac{(x_n(s)-x_m(s))-(x_n(t)-x_m(t))}{s-t}\right| \leq \varepsilon$. Moreover, by uniform convergence, there exists $M \colon (0,\infty) \times [0,1]^2 \to \mathbb{N}$ such that for all $s,t \in [0,1]$ with $s \neq t$, all $\varepsilon \in (0,\infty)$, and all $n \geq M_{\varepsilon,s,t}$ we have $\|x_n - x_\infty\|_{C([0,1],\mathbb{R})} \leq \varepsilon|s-t|$. Thus, we get for all $\varepsilon \in (0,\infty)$, all $s,t \in [0,1]$ with $s \neq t$, and all $n \geq N_\varepsilon$:

$$\begin{aligned}
&|(x_n(s) - x_\infty(s)) - (x_n(t) - x_\infty(t))| \\
&\leq |(x_n(s) - x_{\max\{N_\varepsilon,M_{\varepsilon,s,t}\}}(s)) - (x_n(t) - x_{\max\{N_\varepsilon,M_{\varepsilon,s,t}\}}(t))| \\
&\quad + |x_{\max\{N_\varepsilon,M_{\varepsilon,s,t}\}}(s) - x_\infty(s)| + |x_\infty(t) - x_{\max\{N_\varepsilon,M_{\varepsilon,s,t}\}}(t)| \\
&\leq \varepsilon|s-t| + 2\|x_{\max\{N_\varepsilon,M_{\varepsilon,s,t}\}} - x_\infty\|_{C([0,1],\mathbb{R})} \leq 3\varepsilon|s-t|,
\end{aligned}$$

which establishes that $(x_n)_{n\in\mathbb{N}}$ converges to x_∞ in $(X, \|\cdot\|_{\mathrm{Lip}})$.

(c) Demonstrate that $(Y, \|\cdot\|_{C^1})$ is isometrically embedded in $(X, \|\cdot\|_{\mathrm{Lip}})$ and that Y is closed in $(X, \|\cdot\|_{\mathrm{Lip}})$.

Solution: Observe for every $y \in C^1([0,1],\mathbb{R})$ that, by the fundamental theorem of calculus and compactness of the interval $[0,1]$, y is also Lipschitz continuous. More precisely, for all $s, t \in [0,1]$, we have:

$$|y(s) - y(t)| \leq \left| \int_t^s y'(r)\, dr \right| \leq |s-t| \|y'\|_{C([0,1],\mathbb{R})}.$$

This implies in particular that $\|y\|_{\text{Lip}} \leq \|y\|_{C^1([0,1],\mathbb{R})}$. Since, on the other hand, for every $t \in [0,1]$ there must be a sequence $(s_n)_{n\in\mathbb{N}} \subseteq [0,1] \setminus \{t\}$ with $\lim_{n\to\infty} \frac{y(s_n)-y(t)}{s_n - t} = y'(t)$, we must have that $\|y\|_{C^1([0,1],\mathbb{R})} \leq \|y\|_{\text{Lip}}$. This estabslishes that Y is isometrically embedded in X. For the closedness of Y in X we just notice that Y is complete. Indeed, if $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Y, then (y_n) and (y_n') are Cauchy sequences in $C([0,1],\mathbb{R})$. Thus, there exist $y_\infty, z_\infty \in C([0,1],\mathbb{R})$ such that $\limsup_{n\to\infty} \|y_n - y_\infty\| + \|y_n' - z_\infty\| = 0$. The fundamental theorem of calculus now shows that z_∞ is the derivative of y_∞ as we have for all $s, t \in [0,1]$:

$$y_\infty(t) - y_\infty(s) = \lim_{n\to\infty} [y_n(t) - y_n(s)] = \lim_{n\to\infty} \int_s^t y_n'(r)\, dr = \int_s^t z_\infty(r)\, dr.$$

(Beware: not just this representation, but the representation together with the fact that z_∞ is continuous imply that y_∞ is classically differentiable everywhere.)

3.3. Completion of metric spaces

Let (X, d) be a metric space. A *completion* of (X, d) is a triple $(\mathbb{X}, \delta, \iota)$, where $(\mathbb{X}, \delta)$ is a complete metric space and $\iota\colon X \to \mathbb{X}$ is an isometric embedding with dense image.

(a) Let $(\mathbb{X}, \delta, \iota)$ be a completion of X. Then it satisfies the following universal property: whenever $\phi\colon X \to Y$ is 1-Lipschitz to a complete metric space (Y, d_Y) then there is a unique 1-Lipschitz map $\Phi\colon \mathbb{X} \to Y$ such that $\phi = \Phi \circ \iota$.

Solution: Note that with $\mathbb{X}$ being the closure of $\iota(X)$, we have that for every $\xi \in \mathbb{X}$, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$ satisfying $\limsup_{n\to\infty} \delta(\iota(x_n), \xi) = 0$. If there is a continuous map $\Phi\colon \mathbb{X} \to Y$ with $\phi = \Phi \circ \iota$, then it needs to hold that

$$\Phi(\xi) = \lim_{n\to\infty} \Phi(\iota(x_n)) = \lim_{n\to\infty} \phi(x_n).$$

Hence, we (try to) define $\Phi\colon \mathbb{X} \to Y$ by

$$\Phi(\xi) = \lim_{n\to\infty} \phi(x_n) \quad \text{for all } \xi \in \mathbb{X} \text{ and all } (x_n)_{n\in\mathbb{N}} \subseteq X \text{ with} \limsup_{n\to\infty} \delta(\xi, \iota(x_n)) = 0.$$

If this was well-defined, then we would have for sure that $\phi = \Phi \circ \iota$. It thus remains to show that Φ defined as above is indeed well-defined and 1-Lipschitz. So let $\xi \in \mathbb{X}$ and $(x_n)_{n\in\mathbb{N}} \subseteq X$ be such that $\iota(x_n) \to \xi$ in $(\mathbb{X}, \delta)$ as $n \to \infty$. This implies that $(\iota(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\mathbb{X}, \delta)$. The assumption that $\iota\colon X \to \mathbb{X}$ is an isometry hence implies that $(x_n)_{n\in\mathbb{N}} \subseteq X$ is Cauchy in (X, d). Since ϕ is 1-Lipschitz, the sequence

$(\phi(x_n))_{n\in\mathbb{N}} \subseteq Y$ is Cauchy in (Y, d_Y). As (Y, d_Y) is complete, there exists $y \in Y$ so that $\limsup_{n\to\infty} d_Y(\phi(x_n), y) = 0$. Moreover, note that if $(\tilde{x}_n)_{n\in\mathbb{N}} \subseteq X$ is another sequence with $\limsup_{n\to\infty} \delta(\iota(\tilde{x}_n), \xi) = 0$, then

$$\begin{aligned}
\limsup_{n\to\infty} d_Y(\phi(\tilde{x}_n), y) &\leq \limsup_{n\to\infty} d_Y(\phi(\tilde{x}_n), \phi(x_n)) + \underbrace{\limsup_{n\to\infty} d_Y(\phi(x_n), y)}_{=0} \\
&\leq \limsup_{n\to\infty} \underbrace{d(\tilde{x}_n, x_n)}_{=\delta(\iota(\tilde{x}_n),\iota(x_n))} \\
&\leq \limsup_{n\to\infty} \delta(\iota(\tilde{x}_n), \xi) + \limsup_{n\to\infty} d_Y(\xi, \iota(x_n)) = 0.
\end{aligned}$$

Hence, there indeed exists a map $\Phi\colon \mathbb{X} \to Y$ satisfying for all $\xi \in \mathbb{X}$ and all $(x_n)_{n\in\mathbb{N}} \subseteq X$ with $\limsup_{n\to\infty} \delta(\iota(x_n), \xi) = 0$ that $\limsup_{n\to\infty} d_Y(\Phi(\xi), \phi(x_n)) = 0$. (This implies in particular that $\Phi(\iota(x)) = \phi(x)$ for every $x \in X$.) Finally, for all $\xi_1, \xi_2 \in \mathbb{X}$ and all $(x_n^{(1)})_{n\in\mathbb{N}}, (x_n^{(2)})_{n\in\mathbb{N}}$ satisfying $\iota(x_n^{(1)}) \to \xi_1$ and $\iota(x_n^{(2)}) \to \xi_2$, we obtain

$$\begin{aligned}
&d_Y(\Phi(\xi_1), \Phi(\xi_2)) \\
&\leq \limsup_{n\to\infty} \left[\underbrace{d_Y(\Phi(\xi_1), \phi(x_n^{(1)}))}_{\to 0 \text{ as } n\to\infty} + d_Y(\phi(x_n^{(1)}), \phi(x_n^{(2)})) + \underbrace{d_Y(\phi(x_n^{(2)}), \Phi(\xi_2))}_{\to 0 \text{ as } n\to\infty} \right] \\
&\leq \limsup_{n\to\infty} d(x_n^{(1)}, x_n^{(2)}) = \limsup_{n\to\infty} \delta(\iota(x_n^{(1)}), \iota(x_n^{(2)})) = \delta(\xi^{(1)}, \xi^{(2)}),
\end{aligned}$$

which ascertains the desired Lipschitz property of Φ.

(b) If $(\mathbb{X}_1, \delta_1, \iota_1)$ and $(\mathbb{X}_2, \delta_2, \iota_2)$ are two completions of X, then there exists a unique isometric isomorphism $\psi\colon \mathbb{X}_1 \to \mathbb{X}_2$ such that $\iota_2 = \psi \circ \iota_1$.

Solution: According to (a), there exist a 1-Lipschitz map $\psi\colon \mathbb{X}_1 \to \mathbb{X}_2$ satisfying $\iota_2 = \psi \circ \iota_1$ and a 1-Lipschitz map $\phi\colon \mathbb{X}_2 \to \mathbb{X}_1$ satisfying $\iota_1 = \phi \circ \iota_2$. Hence, we have for all $\xi \in \iota_1(X)$ that $(\phi \circ \psi)(\xi) = \xi$ and for all $\eta \in \iota_2(X)$ that $(\psi \circ \phi)(\eta) = \eta$. Continuity of ψ and ϕ, paired with $\overline{\iota_1(X)} = \mathbb{X}_1$ and $\overline{\iota_2(X)} = \mathbb{X}_2$ ensures that $\psi \circ \phi = \mathrm{id}_{\mathbb{X}_2}$ and $\phi \circ \psi = \mathrm{id}_{\mathbb{X}_1}$. Hence, ψ and ϕ are bijective. Both being 1-Lipschitz, we obtain for all $x_1, y_1 \in \mathbb{X}_1$ and $x_2, y_2 \in \mathbb{X}_2$:

$$\begin{aligned}
\delta_1(x_1, y_1) &= \delta_1(\phi(\psi(x_1)), \phi(\psi(y_1))) \leq \delta_2(\psi(x_1), \psi(y_1)) \leq \delta_1(x_1, y_1) \\
\delta_2(x_2, y_2) &= \delta_2(\psi(\phi(x_2)), \psi(\phi(y_2))) \leq \delta_1(\phi(x_2), \phi(y_2)) \leq \delta_2(x_2, y_2).
\end{aligned}$$

This implies that ϕ and ψ are isometries.

(c) Prove the existence of a completion of (X, d).

Hint: Recall that the space of continuous bounded real-valued functions $C_b(X, \mathbb{R})$ is a Banach space with respect to the norm $\|f\|_\infty = \sup_{x\in X} |f(x)|$. Fix $x_0 \in X$.

For $y \in X$ let $f_y(x) = d(y,x) - d(x_0,x)$. Prove that $\iota(y) = f_y$ defines an isometric embedding $\iota\colon X \to C_b(X,\mathbb{R})$ and put $\mathbb{X} = \overline{\iota(X)}$.

Solution: Note that ι is well-defined, i.e., for every $y \in X$ it holds that f_y is continuous and bounded. Boundedness follows from $|d(y,x) - d(x_0,x)| \leq d(y,x_0)$ for all $x, y \in X$. Continuity follows from $|f_y(x) - f_y(z)| \leq |d(y,x) - d(y,z)| + |d(x_0,x) - d(x_0,z)| \leq 2d(x,z)$ for all $x,y,z \in X$. It remains to show that ι is an isometry. For this, note that for all $x, y_1, y_2 \in X$ it holds that

$$f_{y_1}(x) - f_{y_2}(x) = d(y_1,x) - d(x_0,x) - (d(y_2,x) - d(x_0,x)) = d(y_1,x) - d(y_2,x).$$

The triangle inequality hence implies for all $y_1, y_2 \in X$:

$$\begin{aligned}\|\iota(y_1) - \iota(y_2)\|_{C_b(X,\mathbb{R})} &= \sup_{x\in X} |f_{y_1}(x) - f_{y_2}(x)| \\ &= \sup_{x\in X} |d(y_1,x) - d(y_2,x)| \leq d(y_1,y_2).\end{aligned}$$

Taking into account that $f_{y_1}(y_2) - f_{y_2}(y_2) = d(y_1,y_2)$, we obtain that

$$\|\iota(y_1) - \iota(y_2)\|_{C_b(X,\mathbb{R})} = d(y_1,y_2),$$

which shows that ι is an isometry. Choosing $\mathbb{X} = \overline{\iota(X)}$ completes the proof.

3.4. Compactly supported sequences and their ℓ^∞-completion

Definition. We denote the space of compactly supported sequences by

$$c_c := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \exists N \in \mathbb{N}\ \forall n \geq N:\ x_n = 0\}$$

and the space of sequences converging to zero by

$$c_0 := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \lim_{n\to\infty} x_n = 0\}.$$

(a) Show that $(c_c, \|\cdot\|_{\ell^\infty})$ is *not* complete. What is a completion of this space?

Solution: For every $k \in \mathbb{N}$, let $x^{(k)} = (x_n^{(k)})_{n\in\mathbb{N}} \in c_c$ be given by

$$x_n^{(k)} = \begin{cases} \frac{1}{n} & \text{for } n \leq k, \\ 0 & \text{for } n > k. \end{cases}$$

Then $(x^{(k)})_{k\in\mathbb{N}}$ is a Cauchy sequence in $(c_c, \|\cdot\|_{\ell^\infty})$. Indeed, for every element $y = (y_n)_{n\in\mathbb{N}} \in c_c$, we have:

$$\limsup_{k\to\infty} \|x^{(k)} - y\|_{\ell^\infty} \geq \frac{1}{\min\{n \in \mathbb{N} \mid y_n = 0\}} > 0.$$

(More intuitively speaking, the limit sequence $x^{(\infty)}$ given by $x_n^{(\infty)} = \frac{1}{n}$ for all $n \in \mathbb{N}$ is not in c_c but in $c_0 \setminus c_c$.). We claim that c_0 is a completion of $(c_c, \|\cdot\|_{\ell^\infty})$.

Proof. It suffices to show $c_0 = \overline{c_c}$, where the closure is taken in ℓ^∞ because then, $(c_0, \|\cdot\|_{\ell^\infty})$ is complete as closed subspace of the complete space $(\ell^\infty, \|\cdot\|_{\ell^\infty})$ and $(c_c, \|\cdot\|_{\ell^\infty})$ is clearly densely isometrically embedded.

"$\subseteq$": Let $x = (x_n)_{n\in\mathbb{N}} \in c_0$. Let $(x^{(k)})_{k\in\mathbb{N}}$ be a sequence of sequences $x^{(k)} = (x_n^{(k)})_{n\in\mathbb{N}}$ in c_c given by

$$x_n^k = \begin{cases} x_n & \text{for } n \le k, \\ 0 & \text{for } n > k. \end{cases}$$

Let $\varepsilon > 0$. By assumption, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_n| < \varepsilon$ for every $n \ge N_\varepsilon$.

$$\Rightarrow \ \forall k \ge N_\varepsilon : \quad \|x^{(k)} - x\|_{\ell^\infty} = \sup_{n>k} |0 - x_n| \le \varepsilon.$$

We conclude that $x^{(k)} \to x$ in ℓ^∞ as $k \to \infty$ and since $x \in c_0$ is arbitrary, $c_0 \subseteq \overline{c_c}$.

"$\supseteq$": Let $x = (x_n)_{n\in\mathbb{N}} \in \overline{c_c}$. Then there exists a sequence $(x^{(k)})_{k\in\mathbb{N}}$ of sequences $x^{(k)} = (x_n^{(k)})_{n\in\mathbb{N}} \in c_c$ such that $x^{(k)} \to x$ in ℓ^∞ as $k \to \infty$. Let $\varepsilon > 0$. In particular, there exists $K \in \mathbb{N}$ such that

$$\sup_{n\in\mathbb{N}} |x_n^{(K)} - x_n| = \|x^{(K)} - x\|_{\ell^\infty} < \varepsilon$$

Since $x^{(K)} \in c_c$ there exists $N_0 \in \mathbb{N}$ such that $x_n^{(K)} = 0$ for all $n \ge N_0$. This implies that

$$\forall n \ge N_0 : \quad |x_n| \le \sup_{n \ge N_0} |0 - x_n| < \varepsilon.$$

We conclude that $x_n \to 0$ as $n \to \infty$ which means that $x \in c_0$. □

(b) Prove the strict inclusion

$$\bigcup_{p=1}^{\infty} \ell^p \subsetneq c_0.$$

Solution: If $(x_n)_{n\in\mathbb{N}} \in \ell^p$ for any $p \ge 1$, then necessarily $x_n \to 0$ for $n \to \infty$ by standard facts concerning summable series. Consequently,[1]

$$\bigcup_{p=1}^{\infty} \ell^p \subseteq c_0.$$

[1]Note that by definition $\bigcup_{p=1}^{\infty} \ell^p$ includes ℓ^p for all $p \in \mathbb{N}$ but *not* for $p = \infty$.

The inclusion is strict, since $y = (y_n)_{n\in\mathbb{N}} \in c_0$ given by

$$y_n = \frac{1}{\log(n+1)}$$

has the property that $y \notin \ell^p$ for any $p \geq 1$. Indeed, given any $p \geq 1$ there exists $N_p \in \mathbb{N}$ such that $\log(n+1) \leq n^{\frac{1}{p}}$ for every $n \geq N_p$ which allows the estimate

$$\sum_{n=1}^{\infty}\Big(\frac{1}{\log(n+1)}\Big)^p \geq \sum_{n=N_p}^{\infty}\Big(\frac{1}{n^{\frac{1}{p}}}\Big)^p = \sum_{n=N_p}^{\infty}\frac{1}{n} = \infty.$$

3.5. Operator norms need not be achieved

We consider the space $X = C([-1,1],\mathbb{R})$ with its usual norm $\|\cdot\|_{C([-1,1],\mathbb{R})}$ and define

$$\begin{aligned}\varphi\colon X &\to \mathbb{R}\\ f &\mapsto \int_0^1 f(t)\,dt - \int_{-1}^0 f(t)\,dt.\end{aligned}$$

(a) Show that $\varphi \in L(X,\mathbb{R})$ with $\|\varphi\|_{L(X,\mathbb{R})} \leq 2$.

Solution: Let $\|\cdot\|$ denote the usual sup norm $\|\cdot\|_{C([-1,1],\mathbb{R})}$. The given map $\varphi\colon X \to \mathbb{R}$ is linear by linearity of the integral. Moreover, the fact that

$$|\varphi(f)| \leq \int_0^1 |f(t)|\,dt + \int_{-1}^0 |f(t)|\,dt \leq 2\|f\| \quad \text{for all } f \in X$$

implies

$$\|\varphi\|_{L(X,\mathbb{R})} = \sup_{f\in X\setminus\{0\}} \frac{|\varphi(f)|}{\|f\|} \leq 2.$$

Since φ is linear, continuity follows from boundedness.

(b) Find a sequence $(f_n)_{n\in\mathbb{N}}$ in X such that $\|f_n\|_{C([-1,1],\mathbb{R})} = 1$ for every $n \in \mathbb{N}$ and such that $\varphi(f_n) \to 2$ as $n \to \infty$. This in fact implies $\|\varphi\|_{L(X,\mathbb{R})} = 2$.

Solution: The sign function $f(x) = \frac{x}{|x|}$ is approximated pointwise by the sequence $(f_n)_{n\in\mathbb{N}}$ of functions $f_n \in X$ given by

$$f_n(t) = \begin{cases} -1, & \text{for } -1 \leq t < -\frac{1}{n},\\ nt, & \text{for } -\frac{1}{n} \leq t < \frac{1}{n},\\ 1, & \text{for } \frac{1}{n} \leq t \leq 1.\end{cases}$$

In particular, $\|f_n\|_X = 1$ for every $n \in \mathbb{N}$. Computing the integrals explicitly, or applying the dominated convergence theorem, we have

$$\lim_{n\to\infty} \varphi(f_n) = 2.$$

(c) Prove that there does not exist $f \in X$ with $\|f\|_{C([-1,1],\mathbb{R})} = 1$ and $|\varphi(f)| = 2$.

Solution: Suppose there exists $f \in X$ with $\|f\| = 1$ and $|\varphi(f)| = 2$. Since φ is linear, we may assume $\varphi(f) = 2$, otherwise we replace f by $-f$. Then, the estimates

$$\left|\int_0^1 f(t)\,dt\right| \leq \max_{x\in[-1,1]} |f(x)| = \|f\|_X = 1, \qquad \left|\int_{-1}^0 f(t)\,dt\right| \leq 1,$$

imply by definition of φ that

$$\int_0^1 f(t)\,dt = -\int_{-1}^0 f(t)\,dt = 1. \tag{$*$}$$

Since f is bounded from above by 1 we can conclude from $(*)$ that $f|_{[0,1]} \equiv 1$. In fact, if $f(t^*) < 1$ for some $t^* \in [0,1]$, then – by continuity – $f < 1$ in some neighbourhood of t^* (in $[0,1]$) of f which together with the uniform bound $f \leq 1$ contradicts $(*)$.

Analogously, we conclude $f|_{[-1,0]} \equiv -1$ which (combined with $f|_{[0,1]} \equiv 1$) leads to a contradiction at 0.

3.6. Unbounded map and approximations

As in problem 3.4, we denote the space of compactly supported sequences by

$$c_c := \{(x_n)_{n\in\mathbb{N}} \in \ell^\infty \mid \exists N \in \mathbb{N}\ \forall n \geq N:\ x_n = 0\}$$

endowed with the norm $\|\cdot\|_{\ell^\infty}$. Consider the map

$$\begin{aligned} T\colon c_c &\to c_c \\ (x_n)_{n\in\mathbb{N}} &\mapsto (nx_n)_{n\in\mathbb{N}} \end{aligned}$$

(a) Show that T is not continuous.

Solution: The operation $T\colon (x_n)_{n\in\mathbb{N}} \mapsto (nx_n)_{n\in\mathbb{N}}$ is linear in each entry and therefore linear as map $T\colon c_c \to c_c$. For every $k \in \mathbb{N}$ we define the sequence $e^{(k)} = (e_n^{(k)})_{n\in\mathbb{N}} \in c_c$ by

$$e_n^{(k)} = \begin{cases} 1, & \text{if } n = k, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\|e^{(k)}\|_{\ell^\infty} = 1$ for every $k \in \mathbb{N}$ but $\|Te^{(k)}\|_{\ell^\infty} = k$ is unbounded for $k \in \mathbb{N}$. As unbounded linear map, T is not continuous. (Or, put differently: the sequence $(\frac{e^{(k)}}{k})_{k\in\mathbb{N}} \subseteq c_c$ converges to 0 as $k \to \infty$, but $(T(\frac{e^{(k)}}{k}))_{k\in\mathbb{N}}$ cannot converge to 0 as $k \to \infty$, since $\|T(\frac{e^{(k)}}{k})\|_{\ell^\infty} = 1$ for every $k \in \mathbb{N}$.)

(b) Construct continuous linear maps $T_m : c_c \to c_c$ such that

$$\forall x \in c_c : \quad T_m x \xrightarrow{m\to\infty} Tx.$$

Solution: For every $m \in \mathbb{N}$ we define

$$\begin{aligned} T_m &: c_c \to c_c \\ (x_n)_{n\in\mathbb{N}} &\mapsto (x_1, 2x_2, 3x_3, \ldots, mx_m, 0, 0, \ldots) \end{aligned}$$

Then T_m is linear. $T_m : (c_c, \|\cdot\|_{\ell^\infty}) \to (c_c, \|\cdot\|_{\ell^\infty})$ is also bounded for every (fixed) $m \in \mathbb{N}$ since for every $x = (x_n)_{n\in\mathbb{N}} \in c_c$

$$\|T_m x\| = \sup_{n\in\mathbb{N}} |(T_m x)_n| = \max_{n\in\{1,\ldots,m\}} |nx_n| \le m\|x\|_{\ell^\infty}.$$

Hence, T_m is continuous.

Let $x = (x_n)_{n\in\mathbb{N}} \in c_c$ be fixed. Then there exists $N \in \mathbb{N}$ such that $x_n = 0$ for all $n \ge N$ which implies $T_m x = Tx$ for all $m \ge N$. In particular,

$$T_m x \xrightarrow{m\to\infty} Tx.$$

3.7. Volterra equation

Let $k \in C([0,1]^2, \mathbb{R})$. The Volterra integral operator $T_k : C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ is given by

$$(T_k f)(t) = \int_0^t k(t,s) f(s)\, ds \quad \text{for all } t \in [0,1], f \in C([0,1], \mathbb{R}).$$

(a) Prove that T_k is well-defined and continuous.

Solution: $T_k f$ is well-defined for every $f \in C([0,1], \mathbb{R})$ as k is continuous. T_k is clearly linear and clearly bounded (as k is continuous and $[0,1]^2$ is compact).

(b) For $\lambda \in \mathbb{R}$, let $\|\cdot\|_\lambda : C([0,1], \mathbb{R}) \to [0, \infty)$ be defined by $\|f\|_\lambda = \sup_{t\in[0,1]} e^{-\lambda t}|f(t)|$ for every $f \in C([0,1], \mathbb{R})$. Show that $\|\cdot\|_\lambda$ defines a norm equivalent to the supremum norm on $C([0,1], \mathbb{R})$.

Solution: The fact that for every $\lambda \in \mathbb{R}$ it holds that $0 < \inf_{t\in[0,1]} e^{\lambda t} \le \sup_{t\in[0,1]} e^{\lambda t} < \infty$ implies

$$\left[\inf_{t\in[0,1]} e^{\lambda t}\right] \sup_{t\in[0,1]} |f(t)| \le \|f\|_\lambda \le \left[\sup_{t\in[0,1]} e^{\lambda t}\right] \sup_{t\in[0,1]} |f(t)| \quad \text{for all } \lambda \in \mathbb{R}.$$

(c) Estimate the operator norm of T_k on $(C([0,1],\mathbb{R}), \|\cdot\|_\lambda)$.

Solution: Note that for all $\lambda \in \mathbb{R}$, $f \in C([0,1],\mathbb{R})$ it holds that

$$\begin{aligned}
\|T_k f\|_\lambda &= \sup_{t\in[0,1]} |e^{-\lambda t}(T_k f)(t)| \\
&= \sup_{t\in[0,1]} \left| \int_0^t e^{-\lambda(t-s)} k(t,s) e^{-\lambda s} f(s)\, ds \right| \\
&\leq \sup_{t\in[0,1]} \int_0^t e^{-\lambda(t-s)} \|k\|_{C([0,1]^2,\mathbb{R})} \|f\|_\lambda \, ds \\
&\leq \begin{cases} \|k\|_{C([0,1]^2,\mathbb{R})}\|f\|_\lambda & \lambda = 0, \\ \frac{1}{\lambda}\|k\|_{C([0,1]^2,\mathbb{R})}\|f\|_\lambda & \lambda > 0, \\ \frac{e^{|\lambda|}}{|\lambda|}\|k\|_{C([0,1]^2,\mathbb{R})}\|f\|_\lambda & \lambda < 0. \end{cases}
\end{aligned}$$

(d) Show that for every $g \in C([0,1],\mathbb{R})$ there exists a unique $f \in C([0,1],\mathbb{R})$ satisfying

$$\forall t \in [0,1]\colon \quad f(t) + \int_0^t k(t,s) f(s)\, ds = g(t).$$

Solution: Let $\lambda > 2\|k\|_{C([0,1]^2,\mathbb{R})}$ and consider the map $\Phi\colon X \to X$, given by $\Phi(f) = g - T_k f$ for every $f \in X$. Observe for all $f_1, f_2 \in X$ that

$$\|\Phi(f_1) - \Phi(f_2)\|_\lambda = \|T_k(f_2 - f_1)\|_\lambda \leq \frac{1}{2}\|f_2 - f_1\|_\lambda.$$

Banach's fixed point theorem (cf. also problem 1.6) ensures that there exists a unique $f \in X$ such that $\Phi(f) = f$.

Alternative solution: For (d), which is undoubtedly the goal of (a)–(c), we can argue in a slightly different way by calculating the *spectral radius* of the operator T_k. We claim that for every $n \in \mathbb{N}$ and every $f \in C([0,1],\mathbb{R})$ and $t \in [0,1]$,

$$|(T^n f)(t)| \leq \frac{t^n}{n!}\|k\|^n_{C([0,1]^2,\mathbb{R})}\|f\|_{C([0,1],\mathbb{R})}.$$

We prove this claim by induction. For $n = 1$ we have for all $f \in C([0,1],\mathbb{R})$, $t \in [0,1]$ that

$$|(Tf)(t)| \leq \int_0^t |k(t,s)||f(s)|\, ds \leq t\|k\|_{C^0([0,1]^2,\mathbb{R})}\|f\|_{C([0,1],\mathbb{R})}.$$

Suppose the claim is true for some $n \in \mathbb{N}$. Then, we get for all $f \in C([0,1],\mathbb{R})$, $t \in [0,1]$:

$$\begin{aligned}
|(T^{n+1}f)(t)| &\leq \int_0^t |k(t,s)||(T^n f)(s)|\,ds \\
&\leq \frac{1}{n!}\|k\|^{n+1}_{C([0,1]^2,\mathbb{R})}\|f\|_{C([0,1],\mathbb{R})}\int_0^t s^n\,ds \\
&= \frac{t^{n+1}}{(n+1)!}\|k\|^{n+1}_{C([0,1]^2,\mathbb{R})}\|f\|_{C([0,1],\mathbb{R})}
\end{aligned}$$

which proves the claim. Since $0 \leq t \leq 1$, the claim implies

$$r_T := \lim_{n\to\infty}\|T^n\|^{\frac{1}{n}} \leq \lim_{n\to\infty}\frac{\|k\|_{C([0,1]^2,\mathbb{R})}}{(n!)^{\frac{1}{n}}} = 0.$$

From $r_T = 0$ we conclude that the operator $(I+T) = (I-(-T))$ is invertible with bounded inverse (Satz 2.2.7). The solution to the Volterra equation $f + Tf = g$ is then given by $f = (1+T)^{-1}g$.