4.1. Null and non-null limits

Denote by c the subspace of ℓ^∞ containing all the convergent sequences and let c_0 denote the subspace of sequences converging to 0.

(a) Prove that c is a closed subspace of ℓ^∞.

Solution: Let $(x^{(n)})_{n\in\mathbb{N}} \subseteq c$ be a sequence (of sequences $x^{(n)} = (x_k^{(n)})_{k\in\mathbb{N}} \in c$) which converges in ℓ^∞ to $x^{(\infty)} = (x_k^{(\infty)})_{k\in\mathbb{N}} \in \ell^\infty$. Since we have for every $n \in \mathbb{N}$ that the evaluation functional $\ell^\infty \ni (y_k)_{k\in\mathbb{N}} \mapsto y_n \in \mathbb{R}$ is linear and bounded (with operator norm 1), we may infer that

$$x_k^{(\infty)} = \lim_{n\to\infty} x_k^{(n)} \quad \text{for all } k \in \mathbb{N}.$$

Moreover, due to $(x^{(n)})_{n\in\mathbb{N}}$ converging to $x^{(\infty)}$ in $(\ell^\infty, \|\cdot\|_{\ell^\infty})$ and due to $(x^{(n)})_{n\in\mathbb{N}} \subseteq c$, there exist $N\colon (0,\infty) \to \mathbb{N}$ and $M\colon \mathbb{N}\times(0,\infty) \to \mathbb{N}$ such that

$$\sup_{n\geq N_\varepsilon} \|x^{(n)} - x^{(\infty)}\|_{\ell^\infty} \leq \varepsilon \quad \text{for all } \varepsilon \in (0,\infty)$$

and

$$\sup_{n,m\geq M_{k,\varepsilon}} |x_n^{(k)} - x_m^{(k)}| \leq \varepsilon \quad \text{for all } k \in \mathbb{N}, \varepsilon \in (0,\infty).$$

Thus, we obtain for all $\varepsilon \in (0,\infty)$ that

$$\sup_{n,m\geq M_{N_\varepsilon,\varepsilon}} |x_n^{(\infty)} - x_m^{(\infty)}| \leq \sup_{n,m\geq M_{N_\varepsilon,\varepsilon}} \left(|x_n^{(\infty)} - x_n^{(N_\varepsilon)}| + |x_n^{(N_\varepsilon)} - x_m^{(N_\varepsilon)}| + |x_m^{(N_\varepsilon)} - x_m^{(\infty)}|\right) \leq 3\varepsilon,$$

which proves that $x^{(\infty)}$ is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$ and therefore belongs to c. (By arguing that the linear mapping $c \ni (y_k)_{k\in\mathbb{N}} \mapsto \lim_{k\to\infty} y_k \in \mathbb{R}$ is continuous, we could show that $\lim_{k\to\infty} x_k^{(\infty)} = \lim_{n\to\infty}[\lim_{k\to\infty} x_k^{(n)}]$, but this is not needed for proving closedness of c in ℓ^∞).

(b) Show that c is separable.

Solution: Since $\mathbb{Q}$ is countable and dense in $\mathbb{R}$, it suffices to find a countable set with dense span (instead of a countable dense set) in c for demonstrating separability of c. With regard to the existence of a countable set with dense span, we define the sequences $e^{(1)}, e^{(2)}, e^{(3)}, \ldots, e^{(\infty)} \in c$ as follows:

$$e_n^{(k)} = \begin{cases} 1 & \text{for } k = \infty, \\ 1 & \text{for } k \in \mathbb{N}, n = k, \\ 0 & \text{for } k \in \mathbb{N}, n \neq k, \end{cases}$$

that is, $e^{(\infty)}$ is the sequence constantly equal to 1 and the sequences $e^{(k)}$, $k \in \mathbb{N}$, vanish everywhere except for one entry (namely the k^{th} for $e^{(k)}$) where they equal 1. Clearly, $\{e^{(k)} \mid k \in \mathbb{N} \cup \{\infty\}\} \subseteq c$. Moreover, we claim that $\operatorname{span}\{e^{(k)} \mid k \in \mathbb{N} \cup \{\infty\}\}$ lies dense in c. For this, let $x = (x_k)_{k\in\mathbb{N}} \in c$ be arbitrary. Then it holds for every $n \in \mathbb{N}$ that

$$y^{(n)} := \left(\lim_{k\to\infty} x_k\right) e^{(\infty)} + \sum_{m=1}^{n} (x_m - \lim_{k\to\infty} x_k) e^{(m)} \in \operatorname{span}\{e^{(k)} \mid k \in \mathbb{N} \cup \{\infty\}\}.$$

In addition, note that

$$\limsup_{n\to\infty} \|y^{(n)} - x\|_{\ell^\infty} = \limsup_{n\to\infty} \sup_{m>n} |x_m - \lim_{k\to\infty} x_k| = 0,$$

i.e., $(y^{(n)})_{n\in\mathbb{N}}$ converges in ℓ^∞ to x.

(c) Construct an isomorphism between c and c_0.

Solution: We implicitly already used the isomorphism in the previous exercise. Define the mappings $T\colon c_0 \to c$ and $S\colon c \to c_0$ by

$$\begin{aligned} Tx &= (x_1 + x_{n+1})_{n\in\mathbb{N}} \quad \text{for all } x = (x_n)_{n\in\mathbb{N}} \in c_0, \\ Sx &= (\lim_{n\to\infty} x_n, x_1 - \lim_{n\to\infty} x_n, x_2 - \lim_{n\to\infty} x_n, \ldots) \quad \text{for all } x = (x_n)_{n\in\mathbb{N}} \in c. \end{aligned}$$

T is welldefined as $\lim_{n\to\infty}(Tx)_n = \lim_{n\to\infty}(x_1 + x_{n+1}) = x_1$ for every $x \in c_0$ and S is welldefined as $\lim_{n\to\infty}(x_n - \lim_{k\to\infty} x_k) = 0$ for every $x \in c$. T and S are clearly linear. Moreover, T and S are clearly bounded (with operator norm 2). Finally, the fact that

$$TSx = ((Sx)_1 + (Sx)_{n+1})_{n\in\mathbb{N}} = (\lim_{k\to\infty} x_k + (x_n - \lim_{k\to\infty} x_k))_{n\in\mathbb{N}} = x$$

for all $x = (x_n)_{n\in\mathbb{N}} \in c$ and the fact that

$$\begin{aligned} STx &= (\lim_{n\to\infty}(Tx)_n, (Tx)_1 - \lim_{n\to\infty}(Tx)_n, (Tx)_2 - \lim_{n\to\infty}(Tx)_n, \ldots) \\ &= (\lim_{n\to\infty}(Tx)_n, x_1 + x_2 - \lim_{n\to\infty}(Tx)_n, x_1 + x_3 - \lim_{n\to\infty}(Tx)_n, \ldots) \\ &= (x_1, x_2, x_3, \ldots) = x \end{aligned}$$

for all $x = (x_n)_{n\in\mathbb{N}} \in c_0$ show bijectivity of T and S.

4.2. Baby Riesz representation

Let $p \in [1, \infty)$. Show that φ belongs to the dual space of ℓ^p, (i.e., $\varphi \in (\ell^p)' = L(\ell^p, \mathbb{R})$) if and only if there exists $(f_n)_{n\in\mathbb{N}} \in \ell^q$ such that

$$\varphi((x_n)_{n\in\mathbb{N}}) = \sum_{n=1}^{\infty} f_n x_n \quad \text{for all } (x_n)_{n\in\mathbb{N}} \in \ell^p,$$

where $q \in [1,\infty]$ is such that $\frac{1}{p}+\frac{1}{q}=1$ (with the convention $\frac{1}{\infty}=0$).

Solution: "$\Leftarrow$": Let $(f_n)_{n\in\mathbb{N}} \in \ell^q$. Hölder's inequality ensures for all $x=(x_k)_{k\in\mathbb{N}} \in \ell^p$ that

$$\sum_{k=1}^{\infty} |f_k x_k| \leq \|f\|_{\ell^q}\|x\|_{\ell^p}.$$

In particular, it holds that $\sum_{k=1}^{\infty} f_k x_k$ exists for every $x=(x_k)_{k\in\mathbb{N}} \in \ell^p$ and the mapping $\ell^p \ni x \mapsto \sum_{k=1}^{\infty} f_k x_k \in \mathbb{R}$ is welldefined, linear and continuous.

"$\Rightarrow$": Consider the case $1<p<\infty$, in which case $q=\frac{p}{p-1} \in (1,\infty)$. Let $\varphi \in L(\ell^p,\mathbb{R})$, that is, φ is linear and there exists $C \in [0,\infty)$ satisfying

$$|\varphi(x)| \leq C\|x\|_{\ell^p} \quad \text{for all } x \in \ell^p. \tag{1}$$

For every $n \in \mathbb{N}$, let $e_n := (\delta_{nk})_{k\in\mathbb{N}} \in \ell^p$. Since $\|e_n\|_{\ell^p}=1$ for every $n\in\mathbb{N}$, we obtain by the above inequality that $(\varphi(e_n))_{n\in\mathbb{N}} \subseteq \ell^\infty$ with $\|(\varphi(e_n))_{n\in\mathbb{N}}\|_{\ell^\infty} \leq C$. Moreover, it clearly holds for all $x=(x_k)_{k\in\mathbb{N}} \in \ell^p$ and all $n\in\mathbb{N}$ that

$$\varphi\left(\sum_{k=1}^{n} x_k e_k\right) = \sum_{k=1}^{n} \varphi(e_k) x_k.$$

Moreover, by (1), we have for all $x=(x_k)_{k\in\mathbb{N}} \in \ell^p$ and all $n\in\mathbb{N}$ that

$$\left|\sum_{k=1}^{n} \varphi(e_k) x_k\right| \leq C\left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} \leq C\|x\|_{\ell^p}.$$

Hence, we obtain for every $n\in\mathbb{N}$, by choosing $x^{(n)}=(x_k^{(n)})_{k\in\mathbb{N}} \in \ell^p$ such that

$$x_k^{(n)} = \begin{cases} \varphi(e_k)|\varphi(e_k)|^{\frac{1}{p-1}-1} & \text{if } k \leq n \text{ and } \varphi(e_k) \neq 0, \\ 0 & \text{if } k > n \text{ or } \varphi(e_k)=0, \end{cases}$$

that

$$\sum_{k=1}^{n} |\varphi(e_k)|^{\frac{p}{p-1}} = \varphi(x^{(n)}) \leq C\left(\sum_{k=1}^{n} |\varphi(e_k)|^{\frac{p}{p-1}}\right)^{\frac{1}{p}}.$$

This implies that

$$\left[\sum_{k=1}^{n} |\varphi(e_k)|^{\frac{p}{p-1}}\right]^{\frac{p-1}{p}} \leq C \quad \text{for all } n \in \mathbb{N}.$$

By letting $n \to \infty$, we obtain $(\varphi(e_k))_{k\in\mathbb{N}} \in \ell^{\frac{p}{p-1}}$ with $\|(\varphi(e_k))_{k\in\mathbb{N}}\|_{\ell^{\frac{p}{p-1}}} \leq C$. Since this holds for every C satisfying (1), we have $\|(\varphi(e_k))_{k\in\mathbb{N}}\|_{\ell^{\frac{p}{p-1}}} \leq \|\varphi\|_{L(\ell^p,\mathbb{R})}$. Moreover, since for every $x = (x_k)_{k\in\mathbb{N}} \in \ell^p$ it holds that

$$\limsup_{n\to\infty} \left\| x - \sum_{k=1}^{n} x_k e_k \right\|_{\ell^p} = 0,$$

we obtain by continuity of φ and by Hölder's inequality for every $x = (x_k)_{k\in\mathbb{N}} \in \ell^p$:

$$\varphi(x) = \lim_{n\to\infty} \varphi\left(\sum_{k=1}^{n} x_k e_k\right) = \lim_{n\to\infty} \sum_{k=1}^{n} \varphi(e_k) x_k = \sum_{k=1}^{\infty} \varphi(e_k) x_k.$$

In addition, Hölder's inequality implies

$$|\varphi(x)| \leq \sum_{k=1}^{\infty} |\varphi(e_k) x_k| \leq \|(\varphi(e_k))_{k\in\mathbb{N}}\|_{\ell^{\frac{p}{p-1}}} \|x\|_{\ell^p} \quad \text{for all } x = (x_k)_{k\in\mathbb{N}} \in \ell^p,$$

i.e., $\|\varphi\|_{L(\ell^p,\mathbb{R})} \leq \|(\varphi(e_k))_{k\in\mathbb{N}}\|_{\ell^{\frac{p}{p-1}}}$. The proof in the case $p = 1$ goes along the same lines.

4.3. Infinite matrices

Consider the double sequence $(a_{jk})_{j,k\in\mathbb{N}}$ with $a_{jk} \in \mathbb{R}$ for every $j, k \in \mathbb{N}$.

(a) Let $\sup_{j,k\in\mathbb{N}} |a_{jk}| < \infty$ and let for every $x = (x_k)_{k\in\mathbb{N}} \in \ell^1$ the sequence Ax be given by

$$[Ax]_j = \sum_{k\in\mathbb{N}} a_{jk} x_k \quad \text{for all } j \in \mathbb{N}.$$

Show that this defines a bounded linear map from ℓ^1 to ℓ^∞ (i.e., $A \in L(\ell^1, \ell^\infty)$). Moreover, prove that $\|A\| = \sup_{j,k\in\mathbb{N}} |a_{jk}|$.

Solution: For all $x = (x_k)_{k\in\mathbb{N}} \in \ell^1$ and all $j \in \mathbb{N}$, it clearly holds that

$$|[Ax]_j| \leq \sum_{k\in\mathbb{N}} |a_{jk}||x_k| \leq \sup_{k\in\mathbb{N}} |a_{jk}| \|x\|_{\ell^1} \leq \sup_{j,k\in\mathbb{N}} |a_{jk}| \|x\|_{\ell^1},$$

showing that $Ax \in \ell^\infty$ with $\|Ax\|_{\ell^\infty} \leq \sup_{j,k\in\mathbb{N}} |a_{jk}| \|x\|_{\ell^1}$. Hence, A maps from ℓ^1 to ℓ^∞. Linearity is clear. Moreover, the previous inequality yields $\|A\|_{L(\ell^1,\ell^\infty)} \leq \sup_{j,k\in\mathbb{N}} |a_{jk}|$. For the proof of the converse inequality, let $(j_\varepsilon)_{\varepsilon\in(0,\infty)}, (k_\varepsilon)_{\varepsilon\in(0,\infty)} \subseteq \mathbb{N}$ satisfy for every $\varepsilon \in (0,\infty)$ that $|a_{j_\varepsilon k_\varepsilon}| > \sup_{j,k\in\mathbb{N}} |a_{jk}| - \varepsilon$. Setting $x^{(\varepsilon)} = (\delta_{l k_\varepsilon})_{l\in\mathbb{N}} \in \ell^1$ for every $\varepsilon \in (0,\infty)$, we obtain:

$$\|Ax^{(\varepsilon)}\|_{\ell^\infty} \geq |[Ax^{(\varepsilon)}]_{j_\varepsilon}| = |a_{j_\varepsilon k_\varepsilon}| > \sup_{j,k\in\mathbb{N}} |a_{jk}| - \varepsilon.$$

Since $\|x^{(\varepsilon)}\|_{\ell^1} = 1$ for every $\varepsilon \in (0,\infty)$, this implies $\|A\|_{L(\ell^1,\ell^\infty)} \geq \sup_{j,k\in\mathbb{N}} |a_{j,k}|$.

(b) Let $\sup_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}}|a_{jk}| < \infty$ and define for $x = (x_k)_{k\in\mathbb{N}} \in \ell^\infty$ the sequence Ax as above. Show that this defines a bounded linear map from ℓ^∞ to ℓ^∞ (i.e., $A \in L(\ell^\infty)$)). Moreover, prove that $\|A\| = \sup_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}}|a_{jk}|$.

Solution: For all $x = (x_k)_{k\in\mathbb{N}} \in \ell^\infty$ and all $j \in \mathbb{N}$, it clearly holds that

$$|[Ax]_j| \le \sum_{k\in\mathbb{N}}|a_{jk}||x_k| \le \sum_{k\in\mathbb{N}}|a_{jk}|\|x\|_{\ell^\infty} \le \sup_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}}|a_{jk}|\|x\|_{\ell^\infty},$$

showing that $Ax \in \ell^\infty$ with $\|Ax\|_{\ell^\infty} \le \sup_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}}|a_{jk}|\|x\|_{\ell^\infty}$. Hence, A maps from ℓ^∞ to ℓ^∞. Linearity is clear. Moreover, the previous inequality yields $\|A\|_{L(\ell^\infty,\ell^\infty)} \le \sup_j\sum_{k\in\mathbb{N}}|a_{jk}|$. For the proof of the converse inequality, let $(j_\varepsilon)_{\varepsilon\in(0,\infty)} \subseteq \mathbb{N}$ satisfy for every $\varepsilon \in (0,\infty)$ that $\sum_{k\in\mathbb{N}}|a_{j_\varepsilon k}| > \sup_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}}|a_{jk}| - \varepsilon$. Setting $x^{(\varepsilon)} = (\operatorname{sgn}(a_{j_\varepsilon k}))_{k\in\mathbb{N}} \in \ell^\infty$ for every $\varepsilon \in (0,\infty)$, we obtain:

$$\|Ax^{(\varepsilon)}\|_{\ell^\infty} \ge |[Ax^{(\varepsilon)}]_{j_\varepsilon}| = \sum_{k\in\mathbb{N}}|a_{j_\varepsilon k}| > \sup_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}}|a_{jk}| - \varepsilon.$$

Since $\|x^{(\varepsilon)}\|_{\ell^\infty} \le 1$ for every $\varepsilon \in (0,\infty)$, this implies $\|A\|_{L(\ell^\infty,\ell^\infty)} \ge \sup_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}}|a_{j,k}|$.

4.4. The Fourier coefficients of functions in $L^1([0,2\pi])$

For $f \in L^1([0,2\pi],\mathbb{C})$, we define the k^{th} *Fourier coefficient* to be

$$\hat{f}(k) = \int_0^{2\pi} f(t)e^{-ikt}\,dt$$

and let $\mathcal{F}(f) = (\hat{f}(k))_{k\in\mathbb{Z}}$.

(a) Show that $\mathcal{F}\colon L^1([0,2\pi],\mathbb{C}) \to \ell^\infty(\mathbb{Z},\mathbb{C})$ defines a bounded linear operator.

Solution: For every $f \in L^1([0,2\pi],\mathbb{C})$ and every $k \in \mathbb{Z}$, we have that

$$\left|\int_0^{2\pi} f(t)e^{-ikt}\,dt\right| \le \int_0^{2\pi} |f(t)e^{-ikt}|\,dt = \|f\|_{L^1([0,2\pi],\mathbb{C})},$$

which implies that $\mathcal{F}$ is well-defined and bounded (with operator norm bounded by 1). Linearity is immediate from the linearity of the integral.

(b) Prove the Riemann–Lebesgue lemma, that is, $\limsup_{|k|\to\infty}|\hat{f}(k)| = 0$ for all $f \in L^1([0,2\pi],\mathbb{C})$.

Solution: First, consider $f \in C([0,2\pi],\mathbb{C})$. Being uniformly continuous, there exists $\delta\colon (0,\infty) \to (0,\infty)$ satisfying for all $\varepsilon \in (0,\infty)$ that

$$\Big(x, y \in [0,2\pi] \text{ and } |x-y| < \delta(\varepsilon)\Big) \Rightarrow |f(x) - f(y)| \le \varepsilon.$$

Hence, for every $\varepsilon \in (0,\infty)$ and all $k \in \mathbb{N}$, we have that

$$\left| \int_0^{2\pi} f(t)e^{-ikt}\,dt - \sum_{m=0}^{\lfloor \frac{2\pi}{\delta(\varepsilon)} \rfloor} f(m\delta(\varepsilon)) \int_{m\delta(\varepsilon)}^{\min\{(m+1)\delta(\varepsilon),2\pi\}} e^{-ikt}\,dt \right| \leq 2\pi\varepsilon.$$

Moreover, for every $\varepsilon \in (0,\infty)$, it holds

$$\limsup_{k\to\infty} \left| \sum_{m=0}^{\lfloor \frac{2\pi}{\delta(\varepsilon)} \rfloor} f(m\delta(\varepsilon)) \int_{m\delta(\varepsilon)}^{\min\{(m+1)\delta(\varepsilon),2\pi\}} e^{-ikt}\,dt \right| \leq \limsup_{k\to\infty} \frac{2}{k}\|f\|_{\sup} \left(\left\lfloor \frac{2\pi}{\delta(\varepsilon)} \right\rfloor + 1 \right) = 0.$$

Combining the above two inequalities implies that

$$\limsup_{k\to\infty} \left| \int_0^{2\pi} f(t)e^{-ikt}\,dt \right| \leq 2\pi\varepsilon \quad \text{for all } \varepsilon \in (0,\infty),$$

proving the claim for $f \in C([0,2\pi],\mathbb{C})$. Next assume $f \in L^1([0,2\pi],\mathbb{C})$. Since $C([0,2\pi],\mathbb{C})$ lies dense in $L^1([0,2\pi],\mathbb{C})$, there exists $g_\varepsilon \in C([0,2\pi],\mathbb{C})$ for every $\varepsilon \in (0,\infty)$ such that $\|f - g_\varepsilon\|_{L^1([0,2\pi],\mathbb{C})} \leq \varepsilon$. Hence, we have for every $\varepsilon \in (0,\infty)$ that

$$\limsup_{k\to\infty} \left| \int_0^{2\pi} e^{-ikt} f(t)\,dt \right| \leq \limsup_{k\to\infty} \left| \int_0^{2\pi} e^{-ikt} g_\varepsilon(t)\,dt \right| + \varepsilon = \varepsilon.$$

This completes the proof.

(c) Let $c_0(\mathbb{Z},\mathbb{C}) \subseteq \ell^\infty(\mathbb{Z},\mathbb{C})$ be the closed subspace of sequences converging to zero. Prove that $\mathcal{F}\colon L^1([0,2\pi],\mathbb{C}) \to c_0(\mathbb{Z},\mathbb{C})$ has dense range but is not onto.

Solution: For every $k \in \mathbb{Z}$, $e_k := (\delta_{kn})_{n\in\mathbb{Z}} \in c_0(\mathbb{Z},\mathbb{C})$ lies in $\mathcal{F}(L^1([0,2\pi],\mathbb{C}))$ as $e_k = \mathcal{F}([0,2\pi] \ni t \mapsto e^{ikt} \in \mathbb{C})$. Since $\operatorname{span}\{e_k \colon k \in \mathbb{Z}\}$ is dense in $c_0(\mathbb{Z},\mathbb{C})$ and $\mathcal{F}$ is linear, we can conclude that $\mathcal{F}$ has dense range. If $\mathcal{F}$ was surjective, then the mapping would have to be continuously invertible by the open mapping theorem (note that $\mathcal{F}$ is injective – for example, because $\hat{f}(k) = 0$ for all $k \in \mathbb{Z}$ implies that $\int_0^{2\pi} f(t)p(t)\,dt = 0$ for every trigonometric polynomial p and, by density then also for every continuous 2π-periodic function p, leaving $f = 0$ as the only possibility by the fundamental principle of the calculus of variations). In order to see that this is not the case, consider the functions $(f_n)_{n\in\mathbb{N}} \subseteq L^1([0,2\pi],\mathbb{C})$, given by

$$f_n(t) = \sum_{k=-n}^{n} e^{ikt} \quad \text{for all } t \in [0,2\pi].$$

Note that for all $n \in \mathbb{N}$, $t \in [0,2\pi]$, we have that

$$f_n(t) = \frac{\sin((n+\frac{1}{2})t)}{\sin(\frac{t}{2})}.$$

Thus, we obtain for every $n \in \mathbb{N}$:

$$\begin{aligned}\|f_n\|_{L^1([0,2\pi],\mathbb{C})} &= \int_0^{2\pi} \frac{|\sin((n+\frac{1}{2})t)|}{|\sin(\frac{t}{2})|}\,dt \geq \int_0^{2\pi} (2n+1)\frac{|\sin((n+\frac{1}{2})t)|}{(n+\frac{1}{2})t}\,dt \\ &= 2\int_0^{(2n+1)\pi} \frac{|\sin(t)|}{t}\,dt.\end{aligned}$$

As the latter integral diverges as $n \to \infty$ (it can, for example, be bounded below by a non-zero constant factor times partial sums of a harmonic series), while $\|\mathcal{F}(f_n)\|_{\ell^\infty} = 1$ for every $n \in \mathbb{N}$, $\mathcal{F}$ cannot be continuously invertible.

4.5. Distance to closed subspaces

Let $(X, \|\cdot\|)$ be a normed $\mathbb{R}$-vector space and let $\varphi \in L(X,\mathbb{R})$ be an element of the dual space of X.

(a) Prove for every $x \in X$ that

$$|\varphi(x)| = \|\varphi\|_{L(X,\mathbb{R})} \operatorname{dist}(x, \ker(\varphi)),$$

where $\operatorname{dist}(x, A) = \inf_{v\in A}\|x - v\|$ for $x \in X$ and $\emptyset \neq A \subseteq X$ denotes the distance of the point x to the set A.

Solution: For $\varphi = 0 \in L(X,\mathbb{R})$, the claim is clearly true. Thus, we assume w.l.o.g. that $\varphi \neq 0$. First, observe for every $x \in X$ and all $v \in \ker(\varphi)$ that

$$|\varphi(x)| = |\varphi(x - v)| \leq \|\varphi\|_{L(X,\mathbb{R})}\|x - v\|.$$

Taking the infimum over $v \in \ker(\varphi)$ implies that $|\varphi(x)| \leq \|\varphi\|_{L(X,\mathbb{R})} \operatorname{dist}(x, \ker(\varphi))$ for all $x \in \mathbb{R}$. For the other inequality, note that there exist $(y_\varepsilon)_{\varepsilon\in(0,\infty)} \subseteq X$ satisfying for every $\varepsilon \in (0, \|\varphi\|_{L(X,\mathbb{R})})$ that $\|y_\varepsilon\| \leq 1$ and $|\varphi(y_\varepsilon)| > \|\varphi\|_{L(X,\mathbb{R})} - \varepsilon > 0$. Since it holds for every $\varepsilon \in (0, \|\varphi\|_{L(X,\mathbb{R})})$ that

$$\varphi(x) = \frac{\varphi(x)}{\varphi(y_\varepsilon)}\varphi(y_\varepsilon) = \varphi\left(\frac{\varphi(x)}{\varphi(y_\varepsilon)}y_\varepsilon\right),$$

we may infer that $x - \frac{\varphi(x)}{\varphi(y_\varepsilon)}y_\varepsilon \in \ker(\varphi)$ for every $\varepsilon \in (0, \|\varphi\|_{L(X,\mathbb{R})})$. Hence, we obtain for every $\varepsilon \in (0, \|\varphi\|_{L(X,\mathbb{R})})$:

$$\begin{aligned}\operatorname{dist}(x, \ker(\varphi)) &= \operatorname{dist}\left(\frac{\varphi(x)}{\varphi(y_\varepsilon)}y_\varepsilon, \ker(\varphi)\right) = \frac{|\varphi(x)|}{|\varphi(y_\varepsilon)|}\operatorname{dist}(y_\varepsilon, \ker(\varphi)) \\ &\leq \frac{|\varphi(x)|}{|\varphi(y_\varepsilon)|}\|y_\varepsilon\| \leq \frac{|\varphi(x)|}{\varphi(y_\varepsilon)} \leq \frac{|\varphi(x)|}{\|\varphi\|_{L(X,\mathbb{R})} - \varepsilon}.\end{aligned}$$

Taking limits as $\varepsilon \to 0$ proves the missing inequality. $\square$

Consider now the $\mathbb{R}$-vector space of continuous functions on the real half-line vanishing at ∞, i.e.,

$$C_0([0,\infty),\mathbb{R}) = \left\{ f \in C([0,\infty),\mathbb{R}) \mid \limsup_{t\to\infty} |f(t)| = 0 \right\},$$

equipped with the sup norm $\|\cdot\|_{\sup}$.

(b) Show that $H = \{f \in C_0([0,\infty),\mathbb{R}) \mid \int_0^\infty e^{-s} f(s)\, ds = 0\}$ is a closed subspace of the Banach space $(C([0,\infty),\mathbb{R}), \|\cdot\|_{\sup})$.

Solution: Let $(f_n)_{n\in\mathbb{N}} \subseteq H$ be a sequence converging to $f_\infty \in C([0,\infty),\mathbb{R})$ with respect to the sup norm. Since, consequentially, there exist $N\colon (0,\infty) \to \mathbb{N}$ and $R\colon \mathbb{N} \times (0,\infty) \to [0,\infty)$ satisfying that

$$\sup_{n\geq N_\varepsilon} \|f_n - f_\infty\|_{\sup} \leq \varepsilon \quad \text{for all } \varepsilon \in (0,\infty)$$

and

$$\sup_{t\geq R_{n,\varepsilon}} |f_n(t)| \leq \varepsilon \quad \text{for all } n \in \mathbb{N}, \varepsilon \in (0,\infty),$$

we obtain that

$$\sup_{t\geq R_{N_\varepsilon,\varepsilon}} |f_\infty(t)| \leq \sup_{t\geq R_{N_\varepsilon,\varepsilon}} (|f_{N_\varepsilon(t)}| + \|f_\infty - f_{N_\varepsilon}\|_{\sup}) \leq 2\varepsilon \quad \text{for all } \varepsilon \in (0,\infty),$$

which demonstrates that $f_\infty \in C_0([0,\infty),\mathbb{R})$. Moreover, Hölder's inequality and the fact that $(f_n)_{n\in\mathbb{N}} \subseteq H$ guarantee that

$$\begin{aligned}
\left| \int_0^\infty e^{-s} f_\infty(s)\, ds \right| &= \limsup_{n\to\infty} \left| \int_0^\infty e^{-s} f_\infty(s)\, ds - \int_0^\infty e^{-s} f_n(s)\, ds \right| \\
&\leq \limsup_{n\to\infty} \int_0^\infty e^{-s} |f_\infty(s) - f_n(s)|\, ds \\
&\leq \limsup_{n\to\infty} \|f_\infty - f_n\|_{\sup} = 0.
\end{aligned}$$

Thus, $f_\infty \in H$ and H is closed. Linearity of H is immediate from the linearity of the integral.

(c) Demonstrate that for every $f \in C_0([0,\infty),\mathbb{R}) \setminus H$, there is no $h \in H$ which realizes the distance, i.e., which satisfies $\operatorname{dist}(f,H) = \|f - h\|_{\sup}$.

Solution: Set $X = C_0([0,\infty),\mathbb{R})$. Note that $\varphi\colon X \to \mathbb{R}$, defined by

$$\varphi(f) = \int_0^\infty e^{-s} f(s)\, ds \quad \text{for all } f \in X,$$

defines a continuous linear functional on X (and $H = \ker(\varphi)$). Hence, if there existed $f \in X \setminus H$, $g \in H$ with $\operatorname{dist}(f, H) = \|f - g\|_{\sup}$, then – according to (a), linearity, and the fact that $\varphi(g) = 0$ – we would have

$$\begin{aligned}|\varphi(f-g)| = |\varphi(f)| &= \|\varphi\|_{L(X,\mathbb{R})} \operatorname{dist}(f, \ker(\varphi)) \\ &= \|\varphi\|_{L(X,\mathbb{R})} \operatorname{dist}(f, H) = \|\varphi\|_{L(X,\mathbb{R})} \|f - g\|_{\sup},\end{aligned}$$

in other words, $f - g \neq 0$ would be an element at which the operator norm of φ is realized. But the operator norm of φ is 1 and is not attained in X. Why is the operator norm of φ equal to 1? Note first for all $f \in X$ that

$$|\varphi(f)| \leq \int_0^\infty e^{-s} |f(s)| \, ds \leq \int_0^\infty e^{-s} \|f\|_{\sup} \, ds = \|f\|_{\sup},$$

which implies that $\|\varphi\|_{L(X,\mathbb{R})} \leq 1$. On the other hand, since for every $\alpha \in (0, \infty)$, the function $[0, \infty) \ni s \mapsto e^{-\alpha s} \in \mathbb{R}$ belongs to X and has sup norm 1, we get that

$$\begin{aligned}\|\varphi\|_{L(X,\mathbb{R})} &\geq \varphi([0, \infty) \ni s \mapsto e^{-\alpha s} \in \mathbb{R}) \\ &= \int_0^\infty e^{-(1+\alpha)s} \, ds = \frac{1}{1+\alpha} \quad \text{for all } \alpha \in (0, \infty).\end{aligned}$$

Letting $\alpha \to 0$, we obtain $\|\varphi\|_{L(X,\mathbb{R})} \geq 1$, from which we conclude $\|\varphi\|_{L(X,\mathbb{R})} = 1$. Why is the operator norm of φ not attained? For every $f \in C([0, \infty), \mathbb{R})$ with $\|f\|_{\sup} \leq 1$, we have that

$$1 - \int_0^\infty e^{-s} f(s) \, ds = \int_0^\infty \underbrace{e^{-s}(1 - f(s))}_{\geq 0} \, ds = 0$$

if and only if $1 - f(s) = 0$ for all $s \in [0, \infty)$, i.e., if and only $f(s) = 1$ for all $s \in [0, \infty)$. But the function constantly equal to 1 does not belong to X.