5.1. Closed subspaces

Show that the subspaces

$$U = \{(x_n)_{n\in\mathbb{N}} \in \ell^1 \mid \forall n \in \mathbb{N}: \ x_{2n} = 0\},$$
$$V = \{(x_n)_{n\in\mathbb{N}} \in \ell^1 \mid \forall n \in \mathbb{N}: \ x_{2n-1} = nx_{2n}\}$$

are both closed in $(\ell^1, \|\cdot\|_{\ell^1})$ while the subspace $U \oplus V$ is not closed in $(\ell^1, \|\cdot\|_{\ell^1})$.

Hint. For the second claim, show $c_c \subseteq U \oplus V$. (Recall c_c from problems 3.4 or 3.6.)

Solution: Keep in mind that the convergence of a sequence $(x^{(k)})_{k\in\mathbb{N}} \subseteq \ell^1$ (with $x^{(k)} = (x_n^{(k)})_{n\in\mathbb{N}} \in \ell^1$ for every $k \in \mathbb{N}$) to $x^{(\infty)} = (x_n^{(\infty)})_{n\in\mathbb{N}} \in \ell^1$ entails convergence of the coefficient sequences, that is:

$$\Big(\limsup_{k\to\infty} \|x^{(k)} - x^{(\infty)}\|_{\ell^1} = 0\Big) \Rightarrow \Big(\forall n \in \mathbb{N}: \ \limsup_{k\to\infty} |x_n^{(k)} - x_n^{(\infty)}| = 0\Big).$$

This observation allows to conclude that U and V are closed. Indeed, let $(x^{(k)})_{k\in\mathbb{N}} \subseteq U$ and $(y^{(k)})_{k\in\mathbb{N}} \subseteq V$ (with $x^{(k)} = (x_n^{(k)})_{n\in\mathbb{N}}$ and $y^{(k)} = (y_n^{(k)})_{n\in\mathbb{N}}$ for every $k \in \mathbb{N}$) be converging to $x^{(\infty)} = (x_n^{(\infty)})_{n\in\mathbb{N}} \in \ell^1$ and $y^{(\infty)} = (y_n^{(\infty)})_{n\in\mathbb{N}} \in \ell^1$, respectively. By definition, $x_{2n}^{(k)} = 0$ and $y_{2n-1}^{(k)} = ny_{2n}^{(k)}$ for every $k \in \mathbb{N}$ and every $n \in \mathbb{N}$. According to the above observation,

$$\begin{aligned} x_{2n}^{(\infty)} &= \lim_{k\to\infty} x_{2n}^{(k)} = 0 \quad \text{and} \\ y_{2n-1}^{(\infty)} &= \lim_{k\to\infty} y_{2n-1}^{(k)} = \lim_{k\to\infty} (ny_{2n}^{(k)}) = ny_{2n}^{(\infty)} \end{aligned}$$

for every $n \in \mathbb{N}$. Thus, $x^{(\infty)} \in U$ and $y^{(\infty)} \in V$. This ensures that U and V are closed subspaces of ℓ^1 (linearity of the spaces U and V is considered to be clear).

For proving that $U \oplus V$ is not closed, we show that c_c lies dense in ℓ^1, that $c_c \subseteq U \oplus V$ and that $U \oplus V \subsetneq \ell^1$. With c_c lying dense in ℓ^1, and $U \oplus V$ containing c_c, $U \oplus V$ can only be closed if $U \oplus V = \ell^1$ (which we claim it is not).

Let us start by showing that $c_c \subseteq U \oplus V$. For this, let $x = (x_m)_{m\in\mathbb{N}} \in c_c$ be arbitrary. Then, $x = u + v$ with $u = (u_m)_{m\in\mathbb{N}}$ and $v = (v_m)_{m\in\mathbb{N}}$ given by

$$u_m = \begin{cases} x_m - nx_{m+1}, & \text{if } m = 2n-1, \\ 0, & \text{if } m \text{ is even} \end{cases} \qquad v_m = \begin{cases} nx_{m+1}, & \text{if } m = 2n-1, \\ x_m, & \text{if } m \text{ is even.} \end{cases}$$

The assumption $x \in c_c$ implies $u, v \in c_c \subseteq \ell^1$. Then, $u \in U$ holds by construction and $v \in V$ follows from $v_{2n-1} = nx_{2n-1+1} = nx_{2n} = nv_{2n}$ for every $n \in \mathbb{N}$.

Next we show that c_c lies dense in $(\ell^1, \|\cdot\|_{\ell^1})$. For this, let $x = (x_n)_{n\in\mathbb{N}} \in \ell^1$ be arbitrary. Define $(x^{(k)})_{k\in\mathbb{N}} \subseteq c_c$ (with $x^{(k)} = (x_n^{(k)})_{n\in\mathbb{N}}$ for every $k \in \mathbb{N}$) by setting

$$x_n^{(k)} = \begin{cases} x_n & \text{for } n < k, \\ 0 & \text{for } n \geq k \end{cases}$$

for every $k \in \mathbb{N}$. Then,

$$\limsup_{k\to\infty} \|x^{(k)} - x\|_{\ell^1} = \limsup_{k\to\infty} \left[\sum_{n=k}^{\infty} |x_n|\right] = 0.$$

Finally, we show that $U \oplus V \neq \ell^1$ by counterexample. For this, let $x = (x_m)_{m\in\mathbb{N}}$ be defined as follows:

$$x_m = \begin{cases} 0, & \text{if } m \text{ is odd}, \\ \frac{1}{n^2}, & \text{if } m = 2n. \end{cases}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ we have $x \in \ell^1$. Suppose $x = u + v$ for $u \in U$ and $v \in V$. Then, $u_{2n} = 0$ implies $v_{2n} = x_{2n} = \frac{1}{n^2}$ for every $n \in \mathbb{N}$. By definition of V, we have $v_{2n-1} = n v_{2n} = \frac{1}{n}$ for every $n \in \mathbb{N}$. However, $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ implies $v \notin \ell^1$ which contradicts the definition of V.

This completes the proof that $U \oplus V$ is not closed.

5.2. Vanishing boundary values

Let $X = C([0,1],\mathbb{R})$ and $U = C_0([0,1],\mathbb{R}) := \{f \in C([0,1],\mathbb{R}) \mid f(0) = 0 = f(1)\}$.

(a) Show that U is a closed subspace of X endowed with the norm $\|\cdot\|_X = \|\cdot\|_{C([0,1],\mathbb{R})}$.

Solution: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in U which converges to f in $(X, \|\cdot\|_X)$. Then, since $f_n(0) = 0 = f_n(1)$, we can conclude $f(0) = 0 = f(1)$, i. e., $f \in U$ by passing to the limit $n \to \infty$ in the following inequalities:

$$|f(0)| = |f_n(0) - f(0)| \leq \sup_{x\in[0,1]} |f_n(x) - f(x)| = \|f_n - f\|_X,$$
$$|f(1)| = |f_n(1) - f(1)| \leq \sup_{x\in[0,1]} |f_n(x) - f(x)| = \|f_n - f\|_X.$$

Remark: What was checked here amounts to verifying that evaluation functionals belong to the dual space of X.

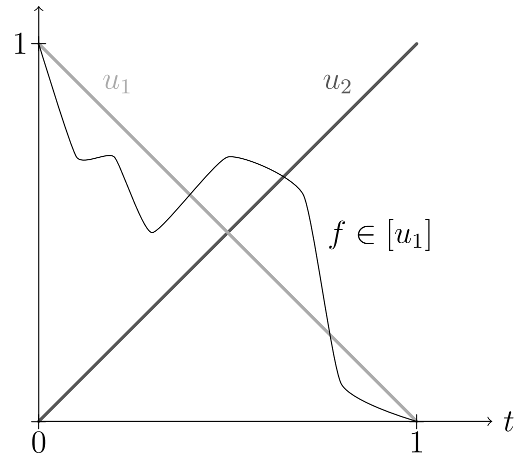

Figure 1: The functions $u_1, u_2 \in X$ and some $f \in [u_1]$.

(b) Compute the dimension of the quotient space X/U and find a basis for X/U.

Solution: Let $u_1, u_2 \in X$ be given by $u_1(t) = 1 - t$ and $u_2(t) = t$. We claim that the equivalence classes $[u_1], [u_2] \in X/U$ form a basis for X/U (and thus, X/U turns out to be a 2-dimensional vector space).

To prove linear independence, let $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1[u_1] + \lambda_2[u_2] = 0 \in X/U$ which means $\lambda_1 u_1 + \lambda_2 u_2 \in U$. This implies by definition

$$\lambda_1 = \lambda_1 u_1(0) + \lambda_2 u_2(0) = 0 = \lambda_1 u_1(1) + \lambda_2 u_2(1) = \lambda_2.$$

To show that $[u_1]$ and $[u_2]$ span X/U, let $[h] \in X/U$ with representative $h \in X$. By evaluation at $t = 0$ and $t = 1$, we conclude

$$\Big(t \mapsto h(t) - h(0)u_1(t) - h(1)u_2(t)\Big) \in U.$$

This implies $[h] = h(0)[u_1] + h(1)[u_2]$ in X/U which proves the claim.

5.3. Continuity of bilinear maps

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces. We consider the space $(X \times Y, \|\cdot\|_{X\times Y})$, where $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$ and a bilinear map $B\colon X \times Y \to Z$.

(a) Show that B is continuous if and only if

$$\exists C > 0 \quad \forall (x,y) \in X \times Y: \quad \|B(x,y)\|_Z \le C\|x\|_X\|y\|_Y. \tag{†}$$

Solution: "$\Leftarrow$": Let $((x_k, y_k))_{k\in\mathbb{N}}$ be a sequence in $X \times Y$ converging to (x_∞, y_∞) in $(X \times Y, \|\cdot\|_{X\times Y})$. By definition, we have for all $k \in \mathbb{N}$ that

$$\|x_k - x_\infty\|_X + \|y_k - y_\infty\|_Y = \|(x_k - x_\infty, y_k - y_\infty)\|_{X\times Y} = \|(x_k, y_k) - (x_\infty, y_\infty)\|_{X\times Y},$$

which yields convergence $x_k \to x_\infty$ in X and $y_k \to y_\infty$ in Y as $k \to \infty$. Since $B\colon X \times Y \to Z$ is bilinear, we have for all $k \in \mathbb{N}$ that

$$\begin{aligned}\|B(x_k, y_k) - B(x_\infty, y_\infty)\|_Z &= \|B(x_k, y_k) - B(x_\infty, y_k) + B(x_\infty, y_k) - B(x_\infty, y_\infty)\|_Z \\ &= \|B(x_k - x_\infty, y_k) + B(x_\infty, y_k - y_\infty)\|_Z \\ &\le \|B(x_k - x_\infty, y_k)\|_Z + \|B(x_\infty, y_k - y)\|_Z.\end{aligned}$$

Using, on one hand, the assumption that $\|B(x, y)\|_Z \le C\|x\|_X\|y\|_Y$ for all $(x, y) \in X \times Y$ and, on the other hand, the fact that convergence of $(y_k)_{k\in\mathbb{N}}$ in $(Y, \|\cdot\|_Y)$ implies that $\sup_{k\in\mathbb{N}}\|y_k\|_Y < \infty$, we conclude

$$\|B(x_k, y_k) - B(x_\infty, y_\infty)\|_Z \le C\|x_\infty - x_k\|_X\|y_k\|_Y + C\|x_\infty\|_X\|y_\infty - y_k\|_Y \xrightarrow{k\to\infty} 0.$$

"$\Rightarrow$": Let B be continuous and assume that (†) does not hold. Hence, there exist sequences $(x_n)_{n\in\mathbb{N}} \subseteq X$ and $(y_n)_{n\in\mathbb{N}} \subseteq Y$ such that

$$\|B(x_n, y_n)\| > n\|x_n\|_X\|y_n\|_Y \quad \text{for all } n \in \mathbb{N}.$$

Note that this implies, in particular, for all $n \in \mathbb{N}$ that $x_n \neq 0$ and $y_n \neq 0$. Thus, we may define sequences $(u_n)_{n\in\mathbb{N}} \subseteq X$ and $(v_n)_{n\in\mathbb{N}} \subseteq Y$ by setting

$$u_n = \frac{x_n}{\sqrt{n}\|x_n\|_X} \quad \text{and} \quad v_n = \frac{y_n}{\sqrt{n}\|y_n\|_Y} \quad \text{for all } n \in \mathbb{N}.$$

By linearity, we have for all $n \in \mathbb{N}$ that $\|B(u_n, v_n)\|_Z > 1$. In view of the fact that $\|(u_n, v_n)\|_{X\times Y} = \|u_n\|_X + \|v_n\|_Y = \frac{2}{\sqrt{n}} \to 0$ as $n \to \infty$, this represents a contradiction to the continuity of the mapping B.

(b) Assume that $(X, \|\cdot\|_X)$ is complete. Assume further that the maps

$$\begin{aligned} X &\to Z \\ x &\mapsto B(x, y') \end{aligned} \qquad\qquad \begin{aligned} Y &\to Z \\ y &\mapsto B(x', y) \end{aligned}$$

are continuous for every $x' \in X$ and $y' \in Y$. Prove that then, (†) holds.

Solution: Let $B_1^Y \subseteq Y$ be the unit ball around the origin in $(Y, \|\cdot\|_Y)$. For every $x \in X$ we have by assumption

$$\sup_{y'\in B_1^Y}\|B(x, y')\|_Z \le \sup_{y'\in B_1^Y}\|y'\|_Y\|B(x, \cdot)\|_{L(Y,Z)} \le \|B(x, \cdot)\|_{L(Y,Z)} < \infty,$$

which means that the maps $(B(\cdot, y'))_{y' \in B_1^Y} \subseteq L(X, Z)$ are pointwise bounded. Since X is assumed to be complete, the uniform boundedness principle (Theorem of Banach–Steinhaus) implies that $(B(\cdot, y'))_{y' \in B_1^Y} \subseteq L(X, Z)$ is bounded, i. e.

$$C := \sup_{y' \in B_1^Y} \|B(\cdot, y')\|_{L(X,Z)} < \infty.$$

From this we conclude for all $x \in X$, $y \in Y$ that

$$\begin{aligned}\|B(x, y)\|_Z &= \|y\|_Y \left\| B\left(x, \frac{y}{\|y\|_Y}\right) \right\|_Z \\ &\leq \|y\|_Y \|x\|_X \left\| B\left(\cdot, \frac{y}{\|y\|_Y}\right) \right\|_{L(X,Z)} \leq C \|x\|_X \|y\|_Y.\end{aligned}$$

5.4. Diverging Fourier series

Prove for every $t \in [0, 2\pi]$ that there exists a continuous 2π-periodic function whose Fourier series does not converge at t.

Solution: Let $X = \{f \in C([0, 2\pi], \mathbb{R}) \mid f(0) = f(2\pi)\}$, equipped with the usual sup norm. For every $m \in \mathbb{N}$ let $s_m \colon X \to \mathbb{R}$ be given by

$$s_m(f) = \frac{1}{2\pi} \sum_{k=-m}^{m} \int_0^{2\pi} f(t) e^{-ikt} \, dt \quad \text{for every } f \in X,$$

that is, $s_m(f)$ is the value of the m^{th} partial Fourier sum of f at 0. Note that, for every $m \in \mathbb{N}$, it holds that s_m is a bounded linear mapping from X to $\mathbb{R}$. Moreover, it holds for all $m \in \mathbb{N}$, $t \in (0, 2\pi)$ that

$$\sum_{k=-m}^{m} e^{ikt} = \frac{e^{i(m+1)t} - e^{-imt}}{e^{it} - 1} = \frac{\sin((m + \frac{1}{2})t)}{\sin(\frac{t}{2})}.$$

Thus, we have for all $m \in \mathbb{N}$, $f \in X$ that

$$s_m(f) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin((m + \frac{1}{2})t)}{\sin(\frac{t}{2})} f(t) \, dt.$$

Note that for every $m \in \mathbb{N}$ it holds that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\sin((m+\frac{1}{2})t)}{\sin(\frac{t}{2})}\right| dt &\geq \int_0^{2\pi}\frac{|\sin((m+\frac{1}{2})t)|}{\pi t}\,dt = \int_0^{(2m+1)\pi}\frac{|\sin(u)|}{\pi u}\,du\\ &\geq \sum_{k=0}^{m}\int_{2k\pi+\frac{\pi}{6}}^{2k\pi+\frac{5\pi}{6}}\frac{|\sin(u)|}{\pi u}\,du\\ &\geq \sum_{k=0}^{2m}\int_{2k\pi+\frac{\pi}{6}}^{2k\pi+\frac{5\pi}{6}}\frac{1}{2\pi u}\,du \geq \sum_{k=0}^{2m}\int_{2k\pi+\frac{\pi}{3}}^{2k\pi+\frac{2\pi}{3}}\frac{1}{2\pi(k+1)}\,du\\ &\geq \sum_{k=0}^{2m}\frac{1}{3(k+1)}.\end{aligned}$$

This implies for every $m \in \mathbb{N}$ that $\|s_m\|_{L(X,\mathbb{R})} \geq \sum_{k=1}^{2m+1}\frac{1}{6k}$. In particular,

$$\sup_{m\in\mathbb{N}}\|s_m\|_{L(X,\mathbb{R})} = \infty.$$

The uniform boundedness principle hence implies – keep in mind that X, equipped with the sup norm, is a Banach space as X is a closed subspace of $C([0,2\pi],\mathbb{R})$ – that there exists $f \in X$ with $\sup_{m\in\mathbb{N}}|s_m(f)| = \infty$. In other words, there exists $f \in X$ such that the partial Fourier sums do not converge at 0 and for every $t \in [0,2\pi]$, the function $[0,2\pi] \ni s \mapsto f(s-t) \in \mathbb{R}$ is a continuous 2π-periodic function whose partial Fourier sums do not converge at t.

5.5. Induced continuity

Let X and Y be Banach spaces, let Z be a metric space, let $T\colon X \to Y$ be linear, let $J\colon Y \to Z$ be injective and continuous, and let $J \circ T\colon X \to Z$ be continuous. Deduce that T is continuous.

Solution: We prove continuity of the linear map $T\colon X \to Y$ by showing that T has a closed graph. For this, let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a sequence such that $x_n \to x_\infty$ in X and $Tx_n \to y_\infty$ in Y. Since $J \circ T$ is continuous, we have

$$\lim_{n\to\infty} J(Tx_n) = \lim_{n\to\infty}(J\circ T)(x_n) = (J\circ T)(x_\infty) = J(Tx_\infty).$$

Moreover, since J is continuous, we have

$$\lim_{n\to\infty} J(Tx_n) = J(y_\infty).$$

As J is injective, we obtain from $J(Tx_\infty) = J(y_\infty)$ that $Tx_\infty = y_\infty$. Thus, T has a closed graph. As X and Y are Banach spaces, the closed graph theorem guarantees continuity of T.

5.6. Projections on closed convex sets

Let H be a Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$, let $C\subseteq H$ be a non-empty, closed and convex set, and let $x\in H$.

(a) Prove that there exists a unique $\xi\in C$ satisfying $\|x-\xi\|=\inf_{y\in C}\|x-y\|$.

Solution: Since C is non-empty and the norm is bounded below, there exists a sequence $(\xi_n)_{n\in\mathbb{N}}\subseteq C$ with

$$\lim_{n\to\infty}\|x-\xi_n\|=\inf_{y\in C}\|x-y\|.$$

The parallelogramm inequality implies for all $m,n\in\mathbb{N}$ that

$$\begin{aligned}2\|\xi_n-x\|^2+2\|\xi_m-x\|^2&=\|\xi_n-\xi_m\|^2+\|\xi_n+\xi_m-2x\|^2\\&=\|\xi_n-\xi_m\|^2+4\|\tfrac{1}{2}(\xi_n+\xi_m)-x\|^2.\end{aligned}$$

The convexity of C implies for all $m,n\in\mathbb{N}$ that $\frac{1}{2}(\xi_n+\xi_m)\in C$. Therefore, we obtain for all $m,n\in\mathbb{N}$:

$$\begin{aligned}\|\xi_n-\xi_m\|^2&=2\|\xi_n-x\|^2+2\|\xi_m-x\|^2-4\|\tfrac{1}{2}(\xi_n+\xi_m)-x\|^2\\&\le 2\|\xi_n-x\|^2+2\|\xi_m-x\|^2-4\inf_{y\in C}\|y-x\|^2.\end{aligned}$$

As $(\xi_n)_{n\in\mathbb{N}}$ is a minimizing sequence, this yields:

$$\limsup_{k\to\infty}\left[\sup_{m,n\ge k}\|\xi_n-\xi_m\|^2\right]\le 0.$$

Thus, $(\xi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and therefore there exists $\xi_\infty\in H$ such that $\xi_n\to\xi_\infty$ as $n\to\infty$. As C is closed, we get that $\xi_\infty\in C$. Moreover do we get by continuity of the norm that

$$\|x-\xi_\infty\|=\lim_{n\to\infty}\|x-\xi_n\|=\inf_{y\in C}\|x-y\|.$$

(b) Prove for all $y\in C$ the following equivalence:

$$\Big(\|x-y\|=\inf_{z\in C}\|x-z\|\Big)\Leftrightarrow\Big(\operatorname{Re}\langle x-y,z-y\rangle\le 0\quad\text{for all } z\in C\Big).$$

Solution: <u>"$\Rightarrow$":</u> Let $y\in C$ be such that $\|x-y\|=\inf_{z\in C}\|x-z\|$. Then it holds – by convexity of C – for every $z\in C$ and all $\alpha\in(0,1)$ that $y+\alpha(z-y)=(1-\alpha)y+\alpha z\in C$ and, therefore $\|x-y-\alpha(z-y)\|\ge\|x-y\|$. Squaring yields for all $\alpha\in(0,1)$ that

$$0\le\|x-y-\alpha(z-y)\|^2-\|x-y\|^2=\alpha^2\|y-z\|^2-2\alpha\operatorname{Re}\langle x-y,z-y\rangle.$$

This implies for all $\alpha \in (0,1)$ and all $z \in C$ that

$$\operatorname{Re}\langle x-y, z-y\rangle \leq \frac{\alpha}{2}\|y-z\|^2.$$

Letting α tend to 0 yields that $\operatorname{Re}\langle x-y, z-y\rangle \leq 0$ for all $z \in C$.

"$\Leftarrow$": Let $y \in C$ be such that $\operatorname{Re}\langle x-y, z-y\rangle \leq 0$ for all $z \in C$. This implies for all $z \in C$ that

$$\begin{aligned}\|x-z\|^2 = \|x-y+y-z\|^2 &= \|x-y\|^2 + 2\operatorname{Re}\langle x-y, y-z\rangle + \|y-z\|^2 \\ &\geq \|x-y\|^2 + \|y-z\|^2 \geq \|x-y\|^2,\end{aligned}$$

as claimed.

5.7. Hardy space

Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ and set

$$\mathcal{H}^2(\mathbb{D}) = \left\{ f\colon \mathbb{D} \to \mathbb{C} \mid f \text{ holomorphic with } \sup_{0\leq r<1} \int_0^{2\pi} |f(re^{it})|^2\, dt < \infty \right\}.$$

(a) Derive a characterization of all the functions $f \in \mathcal{H}^2(\mathbb{D})$ in terms of the coefficients $(a_k(f))_{k\in\mathbb{N}_0} \subseteq \mathbb{C}$ of its power series expansion.

Solution: From complex analysis, we know for every $f \in \mathcal{H}^2(\mathbb{D})$ that $f(z) = \sum_{k=0}^{\infty} a_k(f) z^k$ (for all $z \in \mathbb{D}$), that the convergence radius is at least 1 and that the power series convergences locally uniformly to f. Thus, we obtain for all $r \in (0,1)$ that

$$\begin{aligned}\int_0^{2\pi} |f(re^{it})|^2\, dt &= \int_0^{2\pi} \sum_{k,l=0}^{\infty} a_k(f)\overline{a_l(f)} r^{k+l} e^{it(k-l)}\, dt \\ &= \sum_{k,l=0}^{\infty} 2\pi a_k(f)\overline{a_l(f)} r^{k+l}\delta_{kl} = 2\pi \sum_{k=0}^{\infty} |a_k(f)|^2 r^{2k}.\end{aligned}$$

Hence, $f \in \mathcal{H}^2(\mathbb{D})$ if and only if $((a_k(f))_{k\in\mathbb{N}_0} \in \ell^2(\mathbb{N}_0, \mathbb{C})$.

(b) Demonstrate that, for all $f, g \in \mathcal{H}^2(\mathbb{D})$, the limit

$$\langle f, g\rangle := \lim_{r\to 1} \int_0^{2\pi} f(re^{it})\overline{g(e^{it})}\, \frac{dt}{2\pi}$$

exists and express it in terms of the coefficients $(a_k(f))_{k\in\mathbb{N}_0}, (a_k(g))_{k\in\mathbb{N}_0} \subseteq \mathbb{C}$ of their power series expansions.

Solution: By similar calculations and arguments as above, we obtain for every $r \in (0,1)$ and all $f, g \in \mathcal{H}^2(\mathbb{D})$ that

$$\begin{aligned}\int_0^{2\pi} f(re^{it})\overline{g(re^{it})}\,dt &= \int_0^{2\pi} \sum_{k,l=0}^{\infty} a_k(f)\overline{a_l(g)}r^{k+l}e^{i(k-l)t}\,dt \\ &= \sum_{k,l=0}^{\infty} 2\pi r^{k+l} a_k(f)\overline{a_l(g)}\delta_{kl} = \sum_{k=0}^{\infty} 2\pi r^{k+l} a_k(f)\overline{a_l(g)}.\end{aligned}$$

Since $(a_k(f))_{k\in\mathbb{N}_0}, (a_k(g))_{k\in\mathbb{N}_0} \in \ell^2(\mathbb{N}_0, \mathbb{C})$ by (a), we obtain – based on Cauchy–Schwarz and dominated convergence – for all $f, g \in \mathcal{H}^2(\mathbb{D})$ that

$$\langle f, g\rangle = \sum_{k=0}^{\infty} a_k(f)\overline{a_l(g)}.$$

(c) Prove that $(\mathcal{H}^2(\mathbb{D}), \langle\cdot,\cdot\rangle)$ is a Hilbert space with $(\mathbb{D} \ni z \mapsto z^n \in \mathbb{C})_{n\in\mathbb{N}_0} \subseteq \mathcal{H}^2(\mathbb{D})$ being an orthonormal basis.

Solution: Exercises (a) and (b) above show that $(\mathcal{H}^2(\mathbb{D}), \langle\cdot,\cdot\rangle)$ is isomorphic to $\ell^2(\mathbb{N}_0, \mathbb{C})$ via the isomorphism

$$\mathcal{H}^2(\mathbb{D}) \ni f \mapsto (a_k(f))_{k\in\mathbb{N}_0} \in \ell^2(\mathbb{N}_0, \mathbb{C}).$$

Since $(e_n)_{n\in\mathbb{N}_0} \subseteq \ell^2(\mathbb{N}_0, \mathbb{C})$ with $e_n = (\delta_{nk})_{k\in\mathbb{N}_0}$ is an orthonormal basis of $\ell^2(\mathbb{N}_0, \mathbb{C})$ and since, for every $n \in \mathbb{N}_0$, e_n corresponds to $\mathbb{D} \ni z \mapsto z^n \in \mathbb{C}$, we can conclude that $(\mathbb{D} \ni z \mapsto z^n \in \mathbb{C})_{n\in\mathbb{N}_0}$ is an orthonormal basis of $\mathcal{H}^2(\mathbb{D})$.