6.1. Topological complement

Definition. Let $(X, \|\cdot\|_X)$ be a Banach space. A subspace $U \subseteq X$ is called *topologically complemented* if there is a subspace $V \subseteq X$ such that the linear map I given by

$$I\colon (U \times V, \|\cdot\|_{U\times V}) \to (X, \|\cdot\|_X), \qquad \|(u,v)\|_{U\times V} := \|u\|_X + \|v\|_X,$$
$$(u,v) \mapsto u+v$$

is a continuous isomorphism of normed spaces with continuous inverse. In this case V is said to be a *topological complement* of U.

(a) Prove that $U \subseteq X$ is topologically complemented if and only if there exists a continuous linear map $P\colon X \to X$ with $P \circ P = P$ and image $P(X) = U$.

Solution: "$\Rightarrow$:" Suppose $U \subseteq X$ is topologically complemented by $V \subseteq X$. Then, $I\colon U \times V \to X$ with $(u,v) \mapsto u+v$ is a continuous isomorphism with continuous inverse. We define

$$P_1\colon U \times V \to U \times V, \qquad P := I \circ P_1 \circ I^{-1}\colon X \to X.$$
$$(u,v) \mapsto (u,0)$$

P_1 is linear, bounded since $\|P_1(u,v)\|_{U\times V} = \|u\|_U \le \|(u,v)\|_{U\times V}$ and hence continuous. As composition of linear continuous maps, P is linear and continuous. Moreover,

$$P \circ P = (I \circ P_1 \circ I^{-1}) \circ (I \circ P_1 \circ I^{-1}) = I \circ P_1 \circ P_1 \circ I^{-1} = I \circ P_1 \circ I^{-1} = P,$$
$$P(X) = I(U \times \{0\}) = U.$$

"$\Leftarrow$:" Suppose $U \subseteq X$ allows a continuous linear map $P\colon X \to X$ with $P \circ P = P$ and $P(X) = U$. Let $V := \ker(P)$. Then

$$P \circ (1-P) = P - P = 0 \qquad \Rightarrow (1-P)(X) \subseteq \ker(P) = V. \tag{1}$$

In fact, $(1-P)(X) = V$ since given $v \in V$ we have $v = (1-P)v$. Analogously,

$$(1-P) \circ P = P - P = 0 \qquad \Rightarrow \; U = P(X) \subseteq \ker(1-P). \tag{2}$$

In fact, $U = \ker(1-P)$ since $x - Px = 0$ implies $x = Px \in U$ for all $x \in X$. The claim is that

$$I\colon U \times V \to X$$
$$(u,v) \mapsto u+v$$

is continuous and has a continuous inverse. Continuity of I follows directly from

$$\|I(u,v)\|_X = \|u+v\|_X \le \|u\|_X + \|v\|_X = \|(u,v)\|_{U\times V} \quad \text{for all } u \in U, v \in V.$$

By the assumptions on P, especially (1), the map

$$\begin{aligned}\Phi\colon X &\to U\times V\\ x&\mapsto \big(Px,(1-P)x\big)\end{aligned}$$

is well-defined and continuous. Since $Pu=u$ for all $u\in U$ by (2) and $Pv=0$ for all $v\in V$ by definition of V, we have

$$\begin{aligned}(\Phi\circ I)(u,v)&=\Phi(u+v)=\Big(Pu+Pv,u-Pu+v-Pv\Big)=(u,v).\\ (I\circ\Phi)(x)&=I(Px,(1-P)x)=Px+(1-P)x=x,\end{aligned}$$

which implies that Φ is inverse to I. Consequently, U is topologically complemented.

(b) Show that a topologically complemented subspace must be closed.

Solution: If $U\subseteq X$ is topologically complemented, then (a) implies existence of a continuous map $P\colon X\to X$ with $P=P\circ P$ and $P(X)=U$, which – as we saw in the proof of (a) – implies $\ker(1-P)=U$. Thus, U must be closed as the kernel of the continuous map $1-P$.

Alternatively one might argue that $U=I(U\times\{0\})=(I^{-1})^{-1}(U\times\{0\})$ has to be closed in $(X,\|\cdot\|_X)$ as I is an isomorphism and $U\times\{0\}$ is closed in $(U\times V,\|\cdot\|_{U\times V})$.

Remark. If X is *not* isomorphic to a Hilbert space, then X has closed subspaces which are *not* topologically complemented [Lindenstrauss & Tzafriri. *On the complemented subspaces problem.* (1971)]. An example is $c_0\subseteq\ell^\infty$ but this is not easy to prove.

6.2. Heavily diverging Fourier series

Let $X=\{f\in C([0,2\pi],\mathbb{R})\colon f(0)=f(2\pi)\}$. For $m\in\mathbb{N}_0$ and $f\in X$ we denote the m^{th} partial sum of the Fourier series by S_mf, that is,

$$(S_mf)(t)=\sum_{k=-m}^{m}\left[\frac{1}{2\pi}\int_0^{2\pi}f(s)e^{-iks}\,ds\right]e^{ikt}.$$

This exercise's goal is to prove the existence of a continuous 2π-periodic function whose Fourier series does not converge at uncountably many points. To this end, let $\{t_k\colon k\in\mathbb{N}\}\subseteq[0,2\pi]$ be dense.

(a) Prove that there exists $f_0\in X$ such that $\sup_{m\in\mathbb{N}}|(S_mf_0)(t_n)|=\infty$ for all $n\in\mathbb{N}$.

Solution: By problem 5.4 (*Diverging Fourier series*) – more precisely, by the proposed solution to problem 5.4 – we have for every $n\in\mathbb{N}$ that

$$\sup_{m\in\mathbb{N}_0}\|X\ni f\mapsto (S_mf)(t_n)\in\mathbb{R}\|_{L(X,\mathbb{R})}=\infty.$$

In other words, for every $n \in \mathbb{N}$, the set $G_n \subseteq L(X, \mathbb{R})$ given by

$$G_n = \{X \ni f \mapsto (S_m f)(t_n) \in \mathbb{R} \mid m \in \mathbb{N}_0\}$$

is an unbounded set of continuous linear operators from X to $\mathbb{R}$. By problem 2.4 (*Singularity condensation*), we know that there exists $f_0 \in X$ such that

$$\sup_{m \in \mathbb{N}_0} |(S_m f_0)(t_n)| = \infty \quad \text{for every } n \in \mathbb{N}.$$

(b) Show for every $k \in \mathbb{N}$ that $\{s \in [0, 2\pi] : |(S_m f_0)(s)| \leq k \text{ for all } m \in \mathbb{N}_0\}$ is closed and meagre.

Solution: Since $S_m f_0 \in X$ for every $m \in \mathbb{N}_0$, we have for every $k \in \mathbb{N}$ that

$$\begin{aligned} D_k :=& \{s \in [0, 2\pi] : |(S_m f_0)(s)| \leq k \text{ for all } m \in \mathbb{N}_0\} \\ =& \bigcap_{m \in \mathbb{N}_0} \{s \in [0, 2\pi] : |(S_m f_0)(s)| \leq k\} \end{aligned}$$

is closed as intersection of closed sets. Moreover, since

$$\{t_n : n \in \mathbb{N}\} \subseteq [0, 2\pi] \setminus D_k \quad \text{for every } k \in \mathbb{N},$$

the sets D_k, $k \in \mathbb{N}$, are nowhere dense (and therefore meagre).

(c) Conclude that there is an uncountable subset of $[0, 2\pi]$ on which the Fourier series of f_0 does not converge.

Solution: By (b), the set

$$\begin{aligned} &\left\{s \in [0, 2\pi] : \sup_{m \in \mathbb{N}_0} |(S_m f_0)(s)| < \infty\right\} \\ &= \bigcup_{k \in \mathbb{N}} \{s \in [0, 2\pi] : |(S_m f_0)(s)| \leq k \text{ for all } m \in \mathbb{N}_0\} \end{aligned}$$

is meagre. Hence, the set

$$A = [0, 2\pi] \setminus \left\{s \in [0, 2\pi] : \sup_{m \in \mathbb{N}_0} |(S_m f_0)(s)| < \infty\right\}$$

cannot be meagre as, in that case, $[0, 2\pi]$ would have to be meagre, which is certainly not true according to Baire's theorem. Furthermore, since A is not meagre, A needs to be uncountable. The fact that $\{s \in [0, 2\pi] : (S_m f_0)(s) \text{ does not converge as } m \to \infty\} \subseteq A$ completes the proof.

6.3. The Fundamental Principles Fail for Non-Complete Spaces

Consider the vector space c_c of real sequences $x = (x_n)_{n\in\mathbb{N}}$ with only finitely many non-zero terms (cf. problems 3.4 and 3.6 as well as problem 5.1). Let $\|x\|_{\ell^1} = \sum_{n=1}^{\infty}|x_n|$ and $\|x\|_{\ell^\infty} = \sup_{n\in\mathbb{N}}|x_n|$ be the ℓ^1 and ℓ^∞ norms, respectively.

(a) The family of linear functionals $\varphi_m\colon c_c \to \mathbb{R}$ given by $\varphi_m(x) = mx_m$, $m \in \mathbb{N}$, is pointwise bounded, but not uniformly bounded (in either norm on c_c).

Solution: For every $x = (x_k)_{k\in\mathbb{N}} \in c_c$ it holds that

$$\begin{aligned}\sup_{m\in\mathbb{N}}|\varphi_m(x)| = \sup_{m\in\mathbb{N}}|mx_m| &= \sup_{m\in\mathbb{N}, x_m\neq 0}|mx_m| \\ &\leq \max\{m\in\mathbb{N}\colon x_m\neq 0\}\|x\|_{\ell^\infty} \\ &\leq \max\{m\in\mathbb{N}\colon x_m\neq 0\}\|x\|_{\ell^1}.\end{aligned}$$

Hence, $\{\varphi_m\colon m\in\mathbb{N}\} \subseteq L((c_c, \|\cdot\|_{\ell^1}), \mathbb{R})$ is pointwise bounded and $\{\varphi_m\colon m\in\mathbb{N}\} \subseteq L((c_c, \|\cdot\|_{\ell^\infty}), \mathbb{R})$ is pointwise bounded. But due to

$$\varphi_m((\delta_{km})_{k\in\mathbb{N}}) = m \quad \text{and} \quad \|(\delta_{km})_{k\in\mathbb{N}}\|_{\ell^1} = 1 = \|(\delta_{km})_{k\in\mathbb{N}}\|_{\ell^\infty} \quad \text{for all } m\in\mathbb{N},$$

we get that $\{\varphi_m\colon m\in\mathbb{N}\}$ is neither bounded in $L((c_c, \|\cdot\|_{\ell^1}), \mathbb{R})$ nor in $L((c_c, \|\cdot\|_{\ell^\infty}), \mathbb{R})$.

(b) The identity operator $(c_c, \|\cdot\|_{\ell^1}) \to (c_c, \|\cdot\|_{\ell^\infty})$ is continuous, but not open.

Solution: The inequality

$$\|x\|_{\ell^\infty} \leq \|x\|_{\ell^1} \quad \text{for all } x\in c_c$$

implies that the map $I\colon (c_c, \|\cdot\|_{\ell^1}) \to (c_c, \|\cdot\|_{\ell^\infty})$, given by $I(x) = x$ for every $x \in c_c$, is continuous. Now define for every $m\in\mathbb{N}$ the sequence $x^{(m)} = (x_k^{(m)})_{k\in\mathbb{N}} \in c_c$ by

$$x_k^{(m)} = \begin{cases} \frac{1}{m} & k \leq m \\ 0 & k > m. \end{cases}$$

Note that

$$\|x^{(m)}\|_{\ell^1} = 1 \quad \text{and} \quad \|x^{(m)}\|_{\ell^\infty} = \frac{1}{m} \quad \text{for all } m\in\mathbb{N}.$$

The injectivity of I implies that there is no open ball around 0 in $(c_c, \|\cdot\|_{\ell^\infty})$ which is contained in the image of the open ball of radius 1 around 0 in $(c_c, \|\cdot\|_{\ell^1})$ under I. This proves that I cannot be open.

(c) The identity operator $(c_c, \|\cdot\|_{\ell^\infty}) \to (c_c, \|\cdot\|_{\ell^1})$ has closed graph, but is not continuous.

Solution: Let $J\colon (c_c, \|\cdot\|_{\ell^\infty}) \to (c_c, \|\cdot\|_{\ell^1})$ be given by $J(x) = x$ for every $x \in c_c$. Using the notation from (b), we have that $J = I^{-1}$. The example given in (b) shows that J is not continuous. Let $(x^{(n)})_{n\in\mathbb{N}} \subseteq c_c$ be such that $x^{(n)} \to x^{(\infty)} \in c_c$ in $(c_c, \|\cdot\|_{\ell^\infty})$ as $n \to \infty$ and $J(x^{(n)}) \to y^{(\infty)}$ in $(c_c, \|\cdot\|_{\ell^1})$ as $n \to \infty$. This implies for all $k \in \mathbb{N}$ that

$$y_k^{(\infty)} = \lim_{n\to\infty} J(x^{(n)})_k = \lim_{n\to\infty} x_k^{(n)} = x_k^{(\infty)},$$

which implies $y^{(\infty)} = x^{(\infty)} = J(x^{(\infty)})$. Hence, J has closed graph.

6.4. Zabreiko's Lemma

Let $(X, \|\cdot\|)$ be a $\mathbb{K}$-Banach space (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$), let $p\colon X \to [0,\infty)$ be a *semi-norm* (that is, for all $x, y \in X$, $\lambda \in \mathbb{K}$ it holds that $p(x+y) \le p(x) + p(y)$ and $p(\lambda x) = |\lambda| p(x)$), and assume that

$$p\biggl(\sum_{k=1}^{\infty} x_k\biggr) \le \sum_{k=1}^{\infty} p(x_k) \quad \text{for all } (x_k)_{k\in\mathbb{N}} \subseteq X \text{ for which } \sum_{k=1}^{\infty} x_k \text{ converges.}$$

(a) Demonstrate that there exists $M \in [0,\infty)$ such that

$$p(x) \le M\|x\| \quad \text{for all } x \in X.$$

This is *Zabreiko's lemma. Hint:* Mimick the proof of the open mapping theorem.

Solution: First, observe that $X = \bigcup_{n\in\mathbb{N}} \overline{\{x \in X\colon p(x) \le n\}}$. Since X is complete, Baire's theorem implies that there exist $N \in \mathbb{N}$, $\xi \in X$, $\varepsilon \in (0,\infty)$ such that

$$\{y \in X\colon \|y - \xi\| < \varepsilon\} \subseteq \overline{\{x \in X\colon p(x) \le N\}}.$$

Due to the fact that $p(x) = p(-x)$ for every $x \in X$, we also have that $-\xi + z \in \overline{\{x \in X\colon p(x) \le N\}}$ for every $z \in X$ with $\|z\| < \varepsilon$. Thus, we have for all $z \in X$ with $\|z\| < \varepsilon$ that there exist $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subseteq X$ with $p(x_n) \le N$ and $p(y_n) \le N$ for all $n \in \mathbb{N}$ satisfying that

$$\xi + z = \lim_{n\to\infty} x_n \quad \text{and} \quad -\xi + z = \lim_{n\to\infty} y_n.$$

Consequently, for $z = \frac{1}{2}((\xi+z) + (-\xi+z)) = \lim_{n\to\infty} \frac{1}{2}(x_n + y_n)$ we have because of

$$p\Bigl(\frac{x_n + y_n}{2}\Bigr) = \frac{1}{2}p(x_n + y_n) \le \frac{1}{2}p(x_n) + \frac{1}{2}p(y_n) \le N$$

that $z \in \overline{\{x \in X : p(x) \leq N\}}$. It remains to show that $z \in \{x \in X : p(x) \leq N\}$ for every $z \in X$ with $\|z\| < \varepsilon$. For this, let $z \in X$ with $\|z\| < \varepsilon$ and choose $\delta \in (\|z\|, \varepsilon)$. Moreover, choose $\alpha \in (0,1)$ such that $(1-\alpha)\varepsilon > \delta$. Note that, still, $\|\varepsilon\frac{z}{\delta}\| < \varepsilon$ and therefore, there exists $x_0 \in X$ with $p(x_0) \leq N$ satisfying

$$\left\|\varepsilon\frac{z}{\delta} - x_0\right\| < \alpha\varepsilon.$$

This, in turn, implies that $\|\frac{1}{\alpha}(\varepsilon\frac{z}{\delta} - x_0)\| < \varepsilon$ and, again, there exists $x_1 \in X$ with $p(x_1) \leq N$ satisfying

$$\left\|\frac{\varepsilon\frac{z}{\delta} - x_0}{\alpha} - x_1\right\| < \alpha\varepsilon.$$

Inductively, we obtain $(x_n)_{n\in\mathbb{N}_0} \subseteq X$ satisfying for all $n \in \mathbb{N}_0$ that $p(x_n) \leq N$ and

$$\left\|\varepsilon\frac{z}{\delta} - \sum_{k=0}^{n} \alpha^k x_k\right\| < \alpha^{n+1}\varepsilon.$$

This implies that $\sum_{k=0}^{\infty} \alpha^k x_k$ exists and equals $\varepsilon\frac{z}{\delta}$. The assumptions on p now ascertain

$$p(z) = \frac{\delta}{\varepsilon}p\Big(\varepsilon\frac{z}{\delta}\Big) \leq \frac{\delta}{\varepsilon}\sum_{k=0}^{\infty} p(\alpha^k x_k) = \frac{\delta}{\varepsilon}\sum_{k=0}^{\infty}\alpha^k p(x_k) \leq \frac{\delta}{\varepsilon}\sum_{k=0}^{\infty}\alpha^k N = \frac{\delta}{\varepsilon}\frac{N}{1-\alpha} \leq N.$$

Hence, we obtain for every $z \in X$ with $\|z\| < \varepsilon$ that $p(x) \leq N$. This implies for every $z \in X$ that $p(x) \leq \frac{N}{\varepsilon}\|x\|$.

6.5. Proving everything by Zabreiko's lemma

Recall Zabreiko's lemma from problem 6.4. In this problem we will infer more or less all the fundamental principles from Zabreiko's lemma. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

(a) (*Uniform boundedness principle.*) For a $\mathbb{K}$-Banach space $(X, \|\cdot\|_X)$, a normed $\mathbb{K}$-vector space $(Y, \|\cdot\|_Y)$ and a collection of continuous linear mappings $\mathcal{F} \subseteq L(X,Y)$, prove (by applying Zabreiko's lemma) that

$$\Big(\sup_{T\in\mathcal{F}}\|Tx\|_Y < \infty \quad \text{for every } x \in X\Big) \Rightarrow \sup_{T\in\mathcal{F}}\|T\|_{L(X,Y)} < \infty.$$

Solution: The assumption that $\sup_{T\in\mathcal{F}}\|Tx\|_Y < \infty$ for every $x \in X$ ensures that the function

$$p\colon \begin{cases} X & \to & [0,\infty), \\ x & \mapsto & \sup_{T\in\mathcal{F}}\|Tx\|_Y \end{cases}$$

is indeed well-defined. Moreover, by linearity and by the triangle inequality it holds clearly for all $\lambda \in \mathbb{K}$, $x, y \in X$ that

$$p(\lambda x) = \sup_{T\in\mathcal{F}} \|T(\lambda x)\|_Y = \sup_{T\in\mathcal{F}} \|\lambda Tx\|_Y = |\lambda| \sup_{T\in\mathcal{F}} \|Tx\|_Y = |\lambda| p(x)$$

and

$$\begin{aligned} p(x+y) &= \sup_{T\in\mathcal{F}} \|T(x+y)\|_Y = \sup_{T\in\mathcal{F}} \|Tx+Ty\|_Y \le \sup_{T\in\mathcal{F}} (\|Tx\|_Y + \|Ty\|_Y) \\ &\le \sup_{T\in\mathcal{F}} \|Tx\|_Y + \sup_{T\in\mathcal{F}} \|Ty\|_Y = p(x) + p(y), \end{aligned}$$

that is, p is a semi-norm. Finally, let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a sequence such that $\sum_{n=1}^\infty x_n$ converges. Since every $T \in \mathcal{F}$ is continuous, we have that

$$\begin{aligned} \left\| T\left(\sum_{n=1}^\infty x_n\right)\right\|_Y &= \left\|\sum_{n=1}^\infty Tx_n\right\|_Y = \lim_{N\to\infty} \left\|\sum_{n=1}^N Tx_n\right\|_Y \le \limsup_{N\to\infty} \sum_{n=1}^N \|Tx_n\|_Y \\ &\le \limsup_{N\to\infty} \sum_{n=1}^N p(x_n) = \sum_{n=1}^\infty p(x_n) \quad \text{for all } T \in \mathcal{F}. \end{aligned}$$

This implies that $p(\sum_{n=1}^\infty x_n) \le \sum_{n=1}^\infty p(x_n)$. Now we're in the position to apply Zabreiko's lemma which ensures that there exists $M \in [0,\infty)$ satisfying

$$\sup_{T\in\mathcal{F}} \|Tx\|_Y = p(x) \le M\|x\|_X.$$

This is nothing else than $\sup_{T\in\mathcal{F}} \|T\|_{L(X,Y)} \le M < \infty$, what we intended to prove.

(b) (*Closed graph theorem.*) For $\mathbb{K}$-Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ and a linear map $T\colon X \to Y$, prove (by applying Zabreiko's lemma) that

$$\Big(\text{graph}(T) = \{(x, Tx) \mid x \in X\} \subseteq X \times Y \text{ is closed}\Big) \Rightarrow T \in L(X,Y).$$

Solution: The fact that $Tx \in Y$ for every $x \in X$ ensures that the function

$$p\colon \begin{cases} X & \to & [0,\infty), \\ x & \mapsto & \|Tx\|_Y \end{cases}$$

is well-defined. Linearity and the triangle inequality again ensure for all $\lambda \in \mathbb{K}$, $x, y \in X$ that

- $p(\lambda x) = \|T(\lambda x)\|_Y = \|\lambda Tx\|_Y = |\lambda|\|Tx\|_Y = |\lambda|p(x)$ and
- $p(x+y) = \|T(x+y)\|_Y = \|Tx+Ty\|_Y \le \|Tx\|_Y + \|Ty\|_Y = p(x) + p(y)$.

Hence p is a semi-norm. Next, let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be a sequence satisfying that $\sum_{n=1}^{\infty} x_n$ converges. In the case that $\sum_{n=1}^{\infty}\|Tx_n\|_Y = \infty$, we clearly have that $p(\sum_{n=1}^{\infty} x_n) \leq \sum_{n=1}^{\infty} p(x_n)$. In the case that $\sum_{n=1}^{\infty}\|Tx_n\|_Y < \infty$, the completeness of Y ensures that $\sum_{n=1}^{\infty} Tx_n$ converges in Y. With $\sum_{n=1}^{\infty} x_n = \lim_{N\to\infty}\sum_{n=1}^{N} x_n$ converging in X and $\sum_{n=1}^{\infty} Tx_n = \lim_{N\to\infty}\sum_{n=1}^{N} Tx_n = \lim_{N\to\infty} T(\sum_{n=1}^{N} x_n)$ converging in Y, the closedness of graph(T) ensures that

$$T\left(\sum_{n=1}^{\infty} x_n\right) = \sum_{n=1}^{\infty} Tx_n.$$

Continuity of $\|\cdot\|_Y$ and the triangle inequality hence ensure that

$$\begin{aligned} p\left(\sum_{n=1}^{\infty} x_n\right) &= \left\|T\left(\sum_{n=1}^{\infty} x_n\right)\right\|_Y = \left\|\sum_{n=1}^{\infty} Tx_n\right\|_Y = \lim_{N\to\infty}\left\|\sum_{n=1}^{N} Tx_n\right\|_Y \\ &\leq \limsup_{N\to\infty}\sum_{n=1}^{N}\|Tx_n\|_Y = \lim_{N\to\infty}\sum_{n=1}^{N} p(x_n) = \sum_{n=1}^{\infty} p(x_n). \end{aligned}$$

(c) (*Open mapping theorem.*) For $\mathbb{K}$-Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ and a surjective continuous linear map $T \in L(X, Y)$, prove (by applying Zabreiko's lemma) that T is open.

Solution: Since T is surjective, the function

$$p\colon \begin{cases} Y & \to \quad [0,\infty), \\ y & \mapsto \quad \inf_{x\in X, Tx=y}\|x\|_X \end{cases}$$

is well-defined. Linearity, surjectivity of T, and the triangle inequality again imply for all $\lambda \in \mathbb{K}$, $y, z \in X$ that

$$p(\lambda y) = \inf_{x\in X, Tx=\lambda y}\|x\|_X = \inf_{x\in X, Tx=y}\|\lambda x\|_X = |\lambda|\inf_{x\in X, Tx=y}\|x\|_X = |\lambda| p(y)$$

and

$$\begin{aligned} p(y+z) &= \inf_{x\in X, Tx=y+z}\|x\|_X \leq \inf_{u,v\in X, Tu=y, Tv=z}\|u+v\|_X \\ &\leq \inf_{u,v\in X, Tu=y, Tv=z}(\|u\|_X + \|v\|_X) \\ &= \inf_{u\in X, Tu=y}\|u\|_X + \inf_{v\in X, Tv=z}\|v\|_X = p(y) + p(z). \end{aligned}$$

Thus, p is a semi-norm. Next, let $(y_n)_{n\in\mathbb{N}} \subseteq Y$ be such that $\sum_{n=1}^{\infty} y_n$ converges in Y. In the case that $\sum_{n=1}^{\infty} p(y_n) = \infty$, we clearly have that $p(\sum_{n=1}^{\infty} y_n) \leq \sum_{n=1}^{\infty} p(y_n)$. In the case that $\sum_{n=1}^{\infty} p(y_n) < \infty$, there exist $(x_{n,\varepsilon})_{(n,\varepsilon)\in\mathbb{N}\times(0,\infty)} \subseteq X$ such that

$$Tx_{n,\varepsilon} = y_n \quad \text{and} \quad \|x_{n,\varepsilon}\|_X \leq p(y_n) + 2^{-n}\varepsilon \quad \text{for all } n \in \mathbb{N}, \varepsilon \in (0,\infty).$$

Hence, we obtain for every $\varepsilon \in (0,\infty)$ that

$$\sum_{n=1}^{\infty} \|x_{n,\varepsilon}\|_X \leq \sum_{n=1}^{\infty} (p(y_n) + 2^{-n}\varepsilon) = \sum_{n=1}^{\infty} p(y_n) + \varepsilon < \infty, \tag{3}$$

which – by completeness of X – ensures that

$$\sum_{n=1}^{\infty} x_{n,\varepsilon} \quad \text{converges in } X.$$

Continuity of T makes sure that

$$T\left(\sum_{n=1}^{\infty} x_{n,\varepsilon}\right) = \sum_{n=1}^{\infty} Tx_{n,\varepsilon} = \sum_{n=1}^{\infty} y_n \quad \text{for every } \varepsilon \in (0,\infty).$$

Combining this with (3) implies that

$$p\left(\sum_{n=1}^{\infty} y_n\right) \leq \sum_{n=1}^{\infty} \|x_{n,\varepsilon}\|_X \leq \sum_{n=1}^{\infty} p(y_n) + \varepsilon \quad \text{for all } \varepsilon \in (0,\infty).$$

Letting $\varepsilon \to 0$ shows that the assumptions of Zabreiko's lemma are satisfied. Thus, there exists $M \in (0,\infty)$ satisfying that

$$\inf_{x\in X, Tx=y} \|x\|_X = p(y) \leq M\|y\|_Y \quad \text{for every } y \in Y.$$

This allows to infer that T maps the open unit ball in X to an open set in Y (which, by linearity, is enough for showing that T is an open map). Indeed, for every $y = Tx$ with $x \in X$, $\|x\|_X < 1$, the above inequality implies that for every $z \in Y$ with $\|z-y\|_Y < \frac{1-\|x\|_X}{2M}$, there exists $\xi \in X$ with $\|\xi\|_X < \frac{3}{4}(1-\|x\|_X)$ satisfying $T\xi = z-y$ and, therefore, $T(x+\xi) = z$ and $\|x+\xi\|_X \leq \|x\|_X + \frac{3}{4}(1-\|x\|_X) < 1$.

6.6. Riesz representation theorem for Hilbert spaces

Let $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$ and let $(H, \langle\cdot,\cdot\rangle)$ be $\mathbb{K}$-Hilbert space.

(a) Prove for every $\varphi \in L(H,\mathbb{K})$ (i.e., every φ in the dual space of H) that there exists a unique $v \in H$ such that

$$\varphi(u) = \langle u, v\rangle \quad \text{for every } u \in H.$$

Solution: Let $\varphi \in L(H,\mathbb{K})$. We first prove the existence of an element $v \in H$ s.t. $\varphi(u) = \langle u, v\rangle$ for all $u \in H$. W.l.o.g. we assume that $\varphi \neq 0$ (as the case $\varphi = 0$ is clear). Take $u \in H$ with $\varphi(u) \neq 0$. We know – from problem 5.6, for example – that there exists a unique $x \in \ker(\varphi)$ satisfying $u - x \perp \ker(\varphi)$. As $u \notin \ker(\varphi)$, we have $u - x \neq 0$ and we may define $e := \frac{u-x}{\|u-x\|}$. Note that, for every $w \in H$, it holds that

$$\varphi(w) = \frac{\varphi(w)}{\varphi(e)}\varphi(e) = \varphi\left(\frac{\varphi(w)}{\varphi(e)}e\right),$$

indicating that $w - \frac{\varphi(w)}{\varphi(e)}e \in \ker(\varphi)$. As $e \perp \ker(\varphi)$, this implies

$$\langle w, e\rangle = \left\langle \frac{\varphi(w)}{\varphi(e)}e, e\right\rangle = \frac{\varphi(w)}{\varphi(e)} \quad \text{for all } w \in H,$$

which results in

$$\varphi(w) = \varphi(e)\langle w, e\rangle = \langle w, \overline{\varphi(e)}e\rangle \quad \text{for all } w \in H.$$

This covers the existence part. For uniqueness, note that for all $v_1, v_2 \in H$ satisfying

$$\varphi(w) = \langle w, v_1\rangle = \langle w, v_2\rangle \quad \text{for all } w \in H,$$

we get immediately that

$$\|v_1 - v_2\|^2 = \langle v_1 - v_2, v_1 - v_2\rangle = \langle v_1 - v_2, v_1\rangle - \langle v_1 - v_2, v_2\rangle = \varphi(v_1 - v_2) - \varphi(v_1 - v_2) = 0.$$

Thus, $v_1 = v_2$.

(b) Prove that the map $T\colon H \to L(H, \mathbb{K})$, defined by

$$(Tv)(u) = \langle u, v\rangle \quad \text{for all } u, v \in H,$$

is antilinear, bijective and isometric.

Solution: By the Cauchy–Schwarz inequality, the map T is well-defined and satisfies $\|Tu\|_{L(H,\mathbb{K})} \leq \|u\|$ for all $u \in H$. Clearly, T is antilinear. Moreover, from (a), we know that T is bijective. Finally, as $(Tu)u = \|u\|^2$ for all $u \in H$, it holds for all $u \in H$ that $\|Tu\|_{L(H,\mathbb{K})} = \|u\|$.

6.7. Reproducing kernels

Let S be a set and let H be a $\mathbb{K}$-Hilbert space (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) of functions on S. A *reproducing kernel* for H is a function $k\colon S \times S \to \mathbb{K}$ satisfying for all $t \in S$, $f \in H$ that $k_t = (S \ni s \mapsto k(s,t) \in \mathbb{K}) \in H$ and $f(t) = \langle f, k_t\rangle$.

(a) Prove that a reproducing kernel, if existent, is unique.

Solution: Let $k, l\colon S \times S \to \mathbb{K}$ be reproducing kernels, i.e., satisfy for all $t \in S$, $f \in H$ that $k_t, l_t \in H$ and $f(t) = \langle f, k_t \rangle = \langle f, l_t \rangle$. Consequentially, it holds for all $t \in S$, $f \in H$ that $0 = \langle f, k_t - l_t \rangle$. Since $k_t - l_t \in H$ for every $t \in S$, this implies that $0 = \|k_t - l_t\|^2$ for all $t \in S$. This ensures that $k_t = l_t$ in H for every $t \in S$.

(b) Show that a reproducing kernel exists if and only if, for every $t \in S$, the mapping $H \ni f \mapsto f(t) \in \mathbb{K}$ is continuous.

Solution: "$\Rightarrow$:" If a reproducing kernel exists, then we have for all $t \in S$, $f \in H$ that $|f(t)| = \langle f, k_t \rangle \leq \|f\| \|k_t\|$. That is, for every $t \in S$, the mapping $H \ni f \mapsto f(t) \in \mathbb{K}$ is continuous.

"$\Leftarrow$:" If it holds for every $t \in S$ that $H \ni f \mapsto f(t) \in \mathbb{K}$ is continuous, then – according to the Riesz representation theorem – there exist elements $(h_t)_{t \in S} \subseteq H$ satisfying for every $t \in S$ that

$$f(t) = \langle f, h_t \rangle \quad \text{for all } f \in H.$$

That is, $h\colon S \times S \to \mathbb{K}$, defined by $h(s,t) = h_t(s)$ for all $s, t \in S$, is a reproducing kernel.

(c) Prove that $H = \overline{\operatorname{span}\{k_t \mid t \in S\}}$ if a reproducing kernel exists.

Solution: Let $h \in \operatorname{span}\{k_t \mid t \in S\}^\perp$. This means nothing else but $0 = \langle h, k_t \rangle$ for all $t \in S$. The defining property of the reproducing kernel k now implies that $h(t) = 0$ for all $t \in S$. Hence,

$$\overline{\operatorname{span}\{k_t \mid t \in S\}} = \operatorname{span}\{k_t \mid t \in S\}^{\perp\perp} = \{0\}^\perp = H.$$

(d) Prove that the Hardy space $\mathcal{H}^2(\mathbb{D})$ (cf. problem 5.7) possesses a reproducing kernel and determine the reproducing kernel for $\mathcal{H}^2(\mathbb{D})$.

Solution: Let $z_0 \in \mathbb{D}$ be arbitrary. We know from complex analysis that

$$f(z_0) = \sum_{n=0}^{\infty} a_n(f) z_0^n.$$

The right hand side can – according to problem 5.7(b) – be interpreted as $\mathcal{H}^2(\mathbb{D})$-scalar product of f and the function $k_{z_0}\colon \mathbb{D} \to \mathbb{C}$, defined by

$$k_{z_0}(z) = \sum_{n=0}^{\infty} \overline{z_0}^n z^n \quad \text{for all } z \in \mathbb{D}.$$

Note that k_{z_0} is well-defined on $\mathbb{D}$ due to $|z_0| < 1$. For the same reason, $k_{z_0} \in \mathcal{H}^2(\mathbb{D})$. As a matter of fact, we may rewrite k_{z_0} via

$$k_{z_0}(z) = \frac{1}{1 - \overline{z_0} z} \quad \text{for all } z \in \mathbb{D}.$$