7.1. Finite-dimensional subspaces are topologically complemented

Let $(X, \|\cdot\|_X)$ be a $\mathbb{K}$-Banach space (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) and $U \subseteq X$ a closed subspace. Show that:

(a) If $\dim(U) < \infty$, then U is topologically complemented.

Solution: It is sufficient to construct a projection map P as in Exercise 6.1. Let $e_1, \dots, e_n$ be a basis of the given finite-dimensional subspace $U \subseteq X$ and let $f_1, \dots, f_n \in L(U, \mathbb{K})$ be the associated dual basis, uniquely defined by the conditions

$$f_i(e_j) = \delta_{ij} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else.} \end{cases}$$

From Hahn–Banach's theorem it follows that there exist extensions $F_1, F_2, \dots, F_n \in L(X; \mathbb{K})$ with $\|F_i\|_{L(X,\mathbb{K})} = \|f_i\|_{L(U,\mathbb{K})}$ for every $i \in \{1, 2, \dots, n\}$. We define

$$P \colon X \to X, \quad P(x) = \sum_{i=1}^{n} F_i(x)\, e_i.$$

Then P is linear and continuous, since

$$\|Px\|_X \leq \Big(\sum_{i=1}^{n} \|F_i\|_{L(X,\mathbb{K})} \|e_i\|_X\Big) \|x\|_X \quad \text{for all } x \in X.$$

By construction, $P(X) \subseteq \operatorname{span}\{e_1, \dots, e_n\} = U$. By definition of f_i and F_i we have $P(e_i) = e_i$ for every $i \in \{1, \dots, n\}$. Therefore, $P(X) = U$. Finally, for every $x \in X$,

$$(P \circ P)(x) = P\Big(\sum_{i=1}^{n} F_i(x)\, e_i\Big) = \sum_{i=1}^{n} F_i(x)\, P(e_i) = \sum_{i=1}^{n} F_i(x)\, e_i = P(x).$$

It follows from Exercise 6.1 that U is topologically complemented.

(b) If $\dim(X/U) < \infty$, then U is topologically complemented.

Solution: Denote by $\pi \colon X \to X/U$, $\pi(x) = [x]$ the canonical quotient map. Since $\dim(X/U) = m < \infty$ we can choose $e_1, e_2, \dots, e_m \in X$ such that $[e_1], \dots, [e_m]$ form a basis of X/U. Similar to the above, let $f_1, \dots f_m \in L(X/U, \mathbb{K})$ be the associated dual basis. For every $i \in \{1, 2, \dots, m\}$, set $F_i := f_i \circ \pi \colon X \to \mathbb{K}$. Next, we define

$$P \colon X \to X, \quad P(x) = \sum_{i=1}^{n} F_i(x)\, e_i.$$

Since $F_i(e_j) = f_i(\pi(e_j)) = f_i([e_j]) = \delta_{ij}$ for all $i, j \in \{1, 2, \dots, m\}$, we have $P \circ P = P$ as above. Since $[e_1], \dots, [e_m]$ is a basis for X/U, the representatives $e_1, \dots, e_m$ must be linearly independent in X. Therefore, $P(x) = 0$ implies $F_i(x) = f_i([x]) = 0$ for every $i \in \{1, \dots, n\}$ which in turn implies $[x] = [0]$ or $x \in U$. Conversely, $x \in U$ implies $\pi(x) = [0]$ and therefore $P(x) = 0$. Thus we have shown $\ker(P) = U$. As in Exercise 6.1, we conclude that $(1 - P)$ is a continuous projection onto U which implies that U is topologically complemented.

7.2. Dual spaces of c_0 and c

Recall the ($\mathbb{R}$-vector) spaces

$$c_0 := \Big\{(x_k)_{k\in\mathbb{N}} \in \ell^\infty \;\Big|\; \lim_{k\to\infty} x_k = 0\Big\}, \qquad c := \Big\{(x_k)_{k\in\mathbb{N}} \in \ell^\infty \;\Big|\; \lim_{k\to\infty} x_k \text{ exists}\Big\}.$$

with norm $\|\cdot\|_{\ell^\infty}$ (cf. problems 3.4 and 4.1).

(a) Show that the dual space of $(c_0, \|\cdot\|_{\ell^\infty})$ is *isometrically* isomorphic to $(\ell^1, \|\cdot\|_{\ell^1})$.

Solution: The linear map $\Psi\colon \ell^1 \to (c_0)^*$ given by

$$\Psi(y)(x) = \sum_{n\in\mathbb{N}} x_n y_n,$$

for $x = (x_n)_{n\in\mathbb{N}} \in c_0$ and $y = (y_n)_{n\in\mathbb{N}} \in \ell^1$ is linear and well-defined, since we can estimate

$$|\Psi(y)(x)| \le \sum_{n\in\mathbb{N}} |x_n y_n| \le \|x\|_{\ell^\infty} \|y\|_{\ell^1},$$

and consequently also $\|\Psi(y)\|_{(c_0)^*} \le \|y\|_{\ell^1}$. Let us show that in fact $\|\Psi(y)\|_{(c_0)^*} = \|y\|_{\ell^1}$ for every $y \in \ell^1$: given $y \in \ell^1$ we can apply $\Psi(y)$ to the sequence $x_k = (x_{k,n})_{n\in\mathbb{N}} \in c_0$ given by

$$x_{k,n} = \begin{cases} \frac{y_n}{|y_n|} & \text{if } n \le k \text{ and } y_n \neq 0, \\ 0 & \text{else,} \end{cases}$$

which satisfies $\|x^{(k)}\|_{\ell^\infty} \le 1$ and

$$\lim_{k\to\infty} |\Psi(y)(x_k)| = \lim_{k\to\infty} \sum_{n=1}^{k} |y_n| = \|y\|_{\ell^1},$$

implying that

$$\|\Psi(y)\|_{(c_0)^*} = \sup_{\substack{x\in c_0 \\ \|x\|_{\ell^\infty}\le 1}} |\Psi(y)(x)| \ge \|y\|_{\ell^1}.$$

Therefore, Ψ is an isometry and in particular is injective.

To prove that Ψ is surjective, we show first that every $f \in (c_0)^*$ is determined by its values on the elements $e_k = (e_{k,n})_{n\in\mathbb{N}} \in c_0$, $k \in \mathbb{N}$, where $e_k = (0, \dots, 0, 1, 0, \dots)$ has the 1 at k-th position: in fact, given $x = (x_n)_{n\in\mathbb{N}} \in c_0$, we have

$$\Big\| x - \sum_{k=1}^{N} x_k e_k \Big\|_{\ell^\infty} = \sup_{n>N} |x_n| \xrightarrow{N\to\infty} 0.$$

and so continuity and linearity of f implies

$$f(x) = \lim_{N\to\infty} f\Big(\sum_{k=1}^{N} x_k e_k\Big) = \lim_{N\to\infty} \sum_{k=1}^{N} x_k f(e_k).$$

Given $f \in (c_0)^*$ we claim that $y := (f(e_k))_{k\in\mathbb{N}} \in \ell^1$ and $\Psi(y) = f$. Indeed, for any $N \in \mathbb{N}$

$$\sum_{k=1}^{N} |f(e_k)| = \sum_{k=1}^{\infty} x_{N,k} f(e_k) = f(x_N) \le \|f\|_{(c_0)^*},$$

where $x_N = (x_{N,k})_{k\in\mathbb{N}} \in c_0$ with $\|x_N\|_{\ell^\infty} \le 1$ is defined by

$$x_{N,k} = \begin{cases} \frac{f(e_k)}{|f(e_k)|} & \text{if } k \le N \text{ and } f(e_k) \neq 0, \\ 0 & \text{else.} \end{cases}$$

Since N is arbitrary, we conclude $y \in \ell^1$. Moreover, given any $x = (x_k)_{k\in\mathbb{N}} \in c_0$ and y as above, we have

$$\Psi(y)(x) = \sum_{k\in\mathbb{N}} x_k y_k = \sum_{k\in\mathbb{N}} x_k f(e_k) = f(x)$$

which shows that Ψ is surjective.

(b) To which space is the dual space of $(c, \|\cdot\|_{\ell^\infty})$ isomorphic?

Solution: The dual space of $(c, \|\cdot\|_{\ell^\infty})$ is also isomorphic to $(c_0)^* \cong \ell^1$ but not isometrically. Recall from Problem 4.1 (*Null and non-null limits*) that the maps $S\colon c \to c_0$ and $T\colon c_0 \to c$, given by

$$Sx = \Big(\lim_{n\to\infty} x_n, (x_1 - \lim_{n\to\infty} x_n), (x_2 - \lim_{n\to\infty} x_n), \dots\Big) \quad \text{for all} x = (x_n)_{n\in\mathbb{N}} \in c$$

and

$$T(y) \mapsto \Big((y_2 + y_1), (y_3 + y_1), (y_4 + y_1), \dots\Big) \quad \text{for all } y = (y_n)_{n\in\mathbb{N}} \in c_0$$

respectively, are continuous linear mappings which are inverse to each other. Now define $\Phi\colon c^* \to (c_0)^*$ by

$$\Phi(f) = f \circ T.$$

As composition of linear maps, Φ is linear (Φ is the dual mapping of T). It is also continuous since

$$|(\Phi f)(y)| = |f(Sy)| \le \|f\|_{c^*}\|Sy\|_{\ell^\infty} \le 2\|f\|_{c^*}\|y\|_{\ell^\infty}$$

By the construction above, Φ is bijective with continuous inverse given by $\Phi^{-1}(g) = g \circ S$ for all $g \in (c_0)^*$.

7.3. Banach Limits

Define the shift operator T on (the $\mathbb{R}$-Banach space) $\ell^\infty = \ell^\infty(\mathbb{N}, \mathbb{R})$ by

$$Ty = (y_{n+1})_{n\in\mathbb{N}} \quad \text{for all } y = (y_n)_{n\in\mathbb{N}} \in \ell^\infty.$$

Consider the subspace $X = \{x \in \ell^\infty \mid \exists y \in \ell^\infty \text{ s.t. } x = y - Ty\}$.

(a) The closure of X contains the space of sequences that converge to zero.

Solution: With $e_n = (\delta_{nk})_{k\in\mathbb{N}}$ for every $n \in \mathbb{N}$, note that

$$T\left(\sum_{k=1}^{n} e_k\right) = \sum_{k=1}^{n-1} e_k \quad \text{for every } n \in \mathbb{N},$$

i.e., $e_n = x_n - Tx_n$ with $x_n = \sum_{k=1}^n e_k$ for every $n \in \mathbb{N}$. Thus, X contains $\{e_n \mid n \in \mathbb{N}\}$ and therefore also the space of sequences of finite support. The closure of the latter is the space of null-sequences c_0.

(b) Let c be the constant sequence $c = (1)_{n\in\mathbb{N}}$. Show that $\operatorname{dist}(c, X) = 1$ where $\operatorname{dist}(c, X) = \inf_{x\in X}\|c - x\|_{\ell^\infty}$.

Solution: Surely $d(c, X) \le \|c\|_{\ell^\infty} = 1$. Suppose $x = y - Ty \in X$ is such that $\|c - x\|_{\ell^\infty} = 1 - \varepsilon$ with $\varepsilon \in (0, 1)$. Then $\inf_{n\in\mathbb{N}} x_n \ge \varepsilon$, and as $x = y - Ty$, we deduce from $y_{n+1} = y_n - x_n \le y_n - \varepsilon$ (for all $n \in \mathbb{N}$) the absurdity that $y_n \le y_0 - n\varepsilon$ for all $n \in \mathbb{N}$.

(c) By the Hahn–Banach theorem there is a linear functional $L\colon \ell^\infty \to \mathbb{R}$ such that $L(c) = 1$, $\|L\|_{L(X,\mathbb{R})} = 1$ and $L(x) = 0$ for all $x \in X$. (*Remark:* Indeed, the linear function $l\colon \operatorname{span}\{c\} \to \mathbb{R}$, given by $l(tc) = t$ for every $t \in \mathbb{R}$, is bounded by the restriction of the sublinear function $X \ni x \mapsto \operatorname{dist}(x, X) \in \mathbb{R}$ to $\operatorname{span}\{c\}$, and therefore possesses a linear extension $L\colon X \to \mathbb{R}$ satisfying $|L(x)| \le \operatorname{dist}(x, X)$ for all $x \in X$.)

(i) Show that $L(Ty) = L(y)$ for all $y \in \ell^\infty$.

Solution: Since, for every $y \in \ell^\infty$, it holds that $y - Ty \in X$ and since L vanishes on X, we obtain $L(y - Ty) = 0$ for all $y \in \ell^\infty$, i.e., $L(y) = L(Ty)$ for all $y \in \ell^\infty$.

(ii) Verify that $L(y) \geq 0$ whenever $y \geq 0$ (in the sense that, for $y = (y_n)_{n\in\mathbb{N}} \in \ell^\infty$, it holds that $y_n \geq 0$ for all $n \in \mathbb{N}$) and deduce that $\liminf_{n\to\infty} y_n \leq L(y) \leq \limsup_{n\to\infty} y_n$ for all $y \in \ell^\infty$. It follows that $L(y) = \lim_{n\to\infty} y_n$ whenever y is convergent.

Solution: Suppose $y \geq 0$ is such that $L(y) < 0$. Set $z = c - \frac{y}{\|y\|_{\ell^\infty}}$ and note that $0 \leq z_n \leq 1$ for all $n \in \mathbb{N}$ while $L(z) > 1$, contradicting $\|L\|_{(\ell^\infty)^*} = 1$. Observe that this implies $L(y) \geq L(z)$ whenever $y \geq z$. Let $C = \liminf_{n\to\infty} y_n$ and choose $N \in \mathbb{N}$ so large that $y_n \geq C - \varepsilon$ for all $n \geq N$. Then $L(y) = L(T^N y) \geq C - \varepsilon$ and thus $L(y) \geq \liminf_{n\to\infty} y_n$. Replace y by $-y$ to get the upper bound.

(iii) Find y and z such that $L(yz) \neq L(y)L(z)$.

Solution: Let $y = (\frac{1+(-1)^{n-1}}{2})_{n\in\mathbb{N}}$ (i.e., the sequence 1, 0, 1, 0, ...) and $z = Ty$. Then $c = y + z$, so $L(y) = L(z) = \frac{1}{2}$, while $0 = L(yz)$.

(iv) Show that there is no $z \in \ell^1$ such that $L(y) = \sum_{n=1}^\infty y_n z_n$ for all $y = (y_n)_{n\in\mathbb{N}} \in \ell^\infty$, so L is a functional in $(\ell^\infty)^* \setminus \ell^1$.

Solution: If there was $z = (z_n)_{n\in\mathbb{N}} \in \ell^1$ so that $L(y) = \sum_{n=1}^\infty y_n z_n$ for every $y = (y_n)_{n\in\mathbb{N}} \in \ell^\infty$, then we would get with $e_k = (\delta_{kn})_{n\in\mathbb{N}}$, $k \in \mathbb{N}$, that

$$z_k = L(e_k) = 0 \quad \text{for all } k \in \mathbb{N},$$

a contradiction.

7.4. Inseparable Disjoint Closed Convex Sets

In the Hilbert space $\ell^2 = \ell^2(\mathbb{N}, \mathbb{R})$ of square summable sequences, set $A = \mathbb{R}e_1$ and let

$$B = \left\{ x \in \ell^2 \colon x_1 \geq n \cdot \left| x_n - \frac{1}{n^{2/3}} \right| \text{ for all } n \geq 2 \right\}.$$

(a) Verify that A and B are disjoint, non-empty, closed and convex.

Solution: The set A is a one-dimensional linear subspace, hence it is closed, convex and non-empty. The set B is non-empty because $x = (n^{-2/3})_{n\in\mathbb{N}}$ is an element of B.

Also, B is clearly closed with respect to coordinate-wise convergence, so it is closed in ℓ^2. If $x, y \in B$ and $t \in [0,1]$, then $(1-t)x + ty \in B$ because

$$\begin{aligned} n\left|(1-t)x_n + ty_n - \frac{1}{n^{2/3}}\right| &= n\left|(1-t)(x_n - \frac{1}{n^{2/3}}) + t(y_n - \frac{1}{n^{2/3}})\right| \\ &\leq (1-t)n\left|x_n - \frac{1}{n^{2/3}}\right| + tn\left|y_n - \frac{1}{n^{2/3}}\right| \\ &\leq (1-t)x_1 + ty_1 \quad \text{for all } n \geq 2. \end{aligned}$$

Finally, if $x \in A \cap B$, then $x_1 \geq n^{1/3}$ for all $n \geq 2$, which is impossible.

(b) Prove that $A - B$ is dense in ℓ^2 and conclude that there is no non-zero continuous linear functional on ℓ^2 which separates A from B.

Solution: Let $x \in \ell^2$. Define $(b^{(n)})_{n\in\mathbb{N}} \subseteq \ell^2$ by

$$b_k^{(n)} = \begin{cases} \max\{l\left|x_l + \frac{1}{l^{2/3}}\right| : 2 \leq l \leq n\} & : k = 1, \\ -x_k & : 2 \leq k \leq n, \\ \frac{1}{k^{2/3}} & : k > n, \end{cases}$$

and $(a^{(n)})_{n\in\mathbb{N}} \subseteq \ell^2$ by

$$a_k^{(n)} = \begin{cases} x_1 + b_1^{(n)} & : k = 1, \\ 0 & : k > 1. \end{cases}$$

Then $a \in A$ and $b \in B$ and

$$\begin{aligned} \limsup_{n\to\infty} \|x - (a^{(n)} - b^{(n)})\|_{\ell^2}^2 &= \limsup_{n\to\infty} \sum_{k=n+1}^{\infty} \left|x_k + \frac{1}{k^{2/3}}\right|^2 \\ &\leq 2\limsup_{n\to\infty} \sum_{k>n} |x_k|^2 + 2\limsup_{n\to\infty} \sum_{k>n} \frac{1}{k^{4/3}} = 0. \end{aligned}$$

If φ is continuous and ≥ 0 on $A - B$, then $\varphi \geq 0$ on all of ℓ^2, so $\varphi = 0$ by linearity.

7.5. Strict convexity and uniqueness of the Hahn–Banach extension

(a) (*Ruston's Theorem*) Show that the following properties of a normed $\mathbb{R}$-vector space $(X, \|\cdot\|_X)$ are equivalent:

(i) If $x \neq y$ and $\|x\|_X = 1 = \|y\|_X$ then $\|\frac{x+y}{2}\|_X < 1$.

(ii) If $x \neq 0 \neq y$ and $\|x + y\|_X = \|x\|_X + \|y\|_X$, then $x = \lambda y$ for some $\lambda > 0$.

(iii) If $\varphi \in X^*$ is a nonzero bounded linear functional, then there is at most one $x \in X$ with $\|x\|_X = 1$ such that $\varphi(x) = \|\varphi\|_{X^*}$.

Remark: A normed space is said to be *strictly convex* if any of these properties is satisfied. Point (i) says that the unit sphere contains no non-trivial line segment. Point (ii) says that equality in the triangle inequality only occurs in the trivial situation. Point (iii) says that for $\varphi \neq 0$ the *support hyperplane* $H_\varphi = \{x \in X \colon \varphi(x) = \|\varphi\|_{X^*}\}$ of the unit sphere meets the sphere in at most one point (note that $\inf_{x\in H_\varphi}\|x\|_X = 1$).

Solution: "*(ii)* $\Rightarrow$ *(i)*": Let $x, y \in X$ with $\|x\|_X = 1 = \|y\|_X$ and $x \neq y$. Since $\|x\|_X = \|y\|_X$, this implies that we cannot have that $x = \lambda y$ with $\lambda > 0$. Consequentially, by (ii), it cannot hold that $\|x+y\|_X = \|x\|_X + \|y\|_X$. Since $\|x+y\|_X \leq \|x\|_X + \|y\|_X$ and equality cannot be the case, we infer that $\|x+y\|_X < \|x\|_X + \|y\|_X = 2$.

"*(i)* $\Rightarrow$ *(ii)*": Let $x, y \in X$ satisfy that $\|x+y\|_X = \|x\|_X + \|y\|_X$ where $0 < \|x\|_X \leq \|y\|_X$. Then

$$\begin{aligned}\left\|\frac{x}{\|x\|_X} + \frac{y}{\|y\|_X}\right\|_X &\geq \left\|\frac{x}{\|x\|_X} + \frac{y}{\|x\|_X}\right\|_X - \left\|\frac{y}{\|x\|_X} - \frac{y}{\|y\|_X}\right\|_X \\ &= \frac{1}{\|x\|_X}(\|x\|_X + \|y\|_X) - \|y\|_X\left(\frac{1}{\|x\|_X} - \frac{1}{\|y\|_X}\right) = 2,\end{aligned}$$

whence by $\frac{x}{\|x\|_X} = \frac{y}{\|y\|_X}$ by (i). Thus, (ii) is satisfied with $\lambda = \frac{\|x\|_X}{\|y\|_X}$.

"*(iii)* $\Rightarrow$ *(i)*": Let $x, y \in X$ with $\|x\|_X = \|y\|_X = 1$. Suppose that their midpoint $\frac{x+y}{2}$ has norm 1 as well. By the Hahn–Banach theorem, there exists $\varphi \in X^*$ such that $\|\varphi\|_{X^*} = 1 = \varphi(\frac{x+y}{2}) = \frac{1}{2}(\varphi(x) + \varphi(y))$. As $|\varphi(x)| \leq 1$ and $|\varphi(y)| \leq 1$, it follows that $\varphi(x) = 1 = \varphi(y)$. By (iii), we have $x = y = \frac{x+y}{2}$.

"*(i)* $\Rightarrow$ *(iii)*": Let $\varphi \in X^* \setminus \{0\}$ be a bounded linear functional and let $x, y \in X$ with $\|x\|_X = 1 = \|y\|_X$ and $\varphi(x) = \|\varphi\|_{X^*} = \varphi(y)$. Since $\|\varphi\|_{X^*} \neq 0$, we infer from $\|\varphi\|_{X^*} = \varphi(\frac{x+y}{2}) \leq \|\varphi\|_{X^*}\|\frac{x+y}{2}\|_X$ that $\|\frac{x+y}{2}\|_X \geq 1$. By (i), $x = y$.

(b) For which $p \in [1, \infty]$ is $L^p([0,1], \mathbb{R})$ strictly convex?

Solution: If $1 < p < \infty$, then the convexity properties of the function $\mathbb{R} \ni x \mapsto |x|^p \in \mathbb{R}$ ensure for all $a, b \in \mathbb{R}$ that $|\frac{a+b}{2}|^p \leq \frac{|a|^p}{2} + \frac{|b|^p}{2}$ with equality if and only if $a = b$. This implies for all $f, g \in L^p([0,1], \mathbb{R})$ with $\|f\|_{L^p} = 1 = \|g\|_{L^p}$ that

$$\left\|\frac{f+g}{2}\right\|_{L^p}^p = \int_{[0,1]} \left|\frac{f+g}{2}\right|^p dx \leq \int_{[0,1]} \frac{|f|^p + |g|^p}{2}\, dx = 1$$

with equality if and only if $f = g$ almost everywhere. Clearly, $L^1([0,1], \mathbb{R})$ and $L^\infty([0,1], \mathbb{R})$ are not strictly convex. In $L^1([0,1], \mathbb{R})$ the norm is additive on functions of disjoint support, in $L^\infty([0,1],\mathbb{R})$ on characteristic functions of sets whose intersection has positive measure.

(c) Is $C([0,1],\mathbb{R})$ strictly convex?

Solution: $C([0,1],\mathbb{R})$ is not strictly convex, as one can see, e.g., with $f = ([0,1] \ni x \mapsto 1 \in \mathbb{R})$ and $g = ([0,1] \ni x \mapsto x \in \mathbb{R})$. These functions satisfy $\|f\|_{\sup} = \|g\|_{\sup} = 1$, $f \neq g$, and $\|\frac{f+g}{2}\|_{\sup} = 1$.

(d) If X^* is strictly convex, then every bounded linear functional ψ defined on a subspace U of X has a unique extension Ψ to all of X such that $\|\Psi\|_{X^*} = \|\psi\|_{L(U,\mathbb{R})}$.

Solution: Let $\psi \in L(U,\mathbb{R}) \setminus \{0\}$. The Hahn–Banach theorem ensures that there exists $\Psi \in X^*$ with $\|\Psi\|_{X^*} = \|\psi\|_{L(U,\mathbb{R})}$. Let $\Phi \in X^*$ satisfy that $\Phi|_U = \Psi|_U = \psi$ and $\|\Phi\|_{X^*} = \|\psi\|_{L(U,\mathbb{R})} = \|\Psi\|_{X^*} > 0$. Then $\frac{\Phi+\Psi}{2} \in X^*$ is also an extension of ψ with $\|\frac{\Phi+\Psi}{2}\|_{X^*} = \|\psi\|_{L(U,\mathbb{R})}$. The strict convexity of X^* implies, according to (a), that $\Psi = \Phi$.

7.6. Another application of the Hahn–Banach theorem

Let $(X, \|\cdot\|_X)$ be a normed $\mathbb{K}$-vector space (with $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$), let $(x_j)_{j\in\mathbb{N}} \subseteq X$ be a sequence of points X, let $\gamma \in [0,\infty)$, and let $(\alpha_j)_{j\in\mathbb{N}} \subseteq \mathbb{K}$ be a sequence. Prove that the following are equivalent:

(i) There exists a functional $l \in X^*$ satisfying

$$\|l\|_{X^*} \leq \gamma \quad \text{and} \quad l(x_j) = \alpha_j \quad \text{for all } j \in \mathbb{N}.$$

(ii) It holds that

$$\left|\sum_{j=1}^n \beta_j \alpha_j\right| \leq \gamma \left\|\sum_{j=1}^n \beta_j x_j\right\|_X \qquad \text{for all } n \in \mathbb{N} \text{ and } (\beta_j)_{j=1}^n \subseteq \mathbb{K}.$$

Solution: $\Rightarrow$: A direct calculation shows for all $n \in \mathbb{N}$ and all $(\beta_j)_{j=1}^n \subseteq \mathbb{K}$ that

$$\left|\sum_{j=1}^n \beta_j \alpha_j\right| = \left|\sum_{j=1}^n \beta_j l(x_j)\right| = \left|l\left(\sum_{j=1}^n \beta_j x_j\right)\right| \leq \|l\|_{X^*} \left\|\sum_{j=1}^n \beta_j x_j\right\|_X .$$

$\Leftarrow$: Let $U = \operatorname{span}\{x_j : j \in \mathbb{N}\}$. Let $\mathfrak{l} \colon U \to \mathbb{K}$ be given by

$$\mathfrak{l}\left(\sum_{k=1}^n \beta_j x_j\right) = \sum_{k=1}^n \beta_j \alpha_j \quad \text{for all } n \in \mathbb{N}, (\beta_j)_{j=1}^n \subseteq \mathbb{K}.$$

The mapping $\mathfrak{l}$ is clearly linear (if it is well-defined at all). It is well-defined, though, since $\sum_{k=1}^{n} \beta_j x_j = 0$ for $n \in \mathbb{N}$, $(\beta_j)_{j=1}^{n} \subseteq \mathbb{K}$ implies that $\sum_{k=1}^{n} \beta_j \alpha_j = 0$. By assumption, $\|\mathfrak{l}\|_{L(U,\mathbb{K})} \leq \gamma$. The Hahn–Banach theorem guarantees the existence of $l \in X^*$ with $\|l\|_{X^*} \leq \gamma$ and $l|_U = \mathfrak{l}$ (and therefore, in particular, $l(x_j) = \alpha_j$ for all $j \in \mathbb{N}$).