8.1. Strict convexity and extremal points

A normed space $(X, \|\cdot\|)$ with $X \neq \{0\}$ is strictly convex (cf. Problem 7.5) if and only if the unit sphere $S := \{x \in X : \|x\| = 1\}$ is equal to the set of extremal points of the closed unit ball $B := \{x \in X : \|x\| \leq 1\}$.

Solution: "$\Rightarrow$": Suppose that $(X, \|\cdot\|)$ is strictly convex. We need to show that $\mathrm{ex}(B) = S$. Note that no point in $B \setminus S$ can be extremal, since every point in $B \setminus S$ is the center of a small ball contained in B. That is, $\mathrm{ex}(B) \subseteq S$. Thus, we only need to prove that $S \subseteq \mathrm{ex}(B)$. For this, let $x \in S$, $y_1, y_2 \in B$ and $\lambda \in (0,1)$ be such that $x = \lambda y_1 + (1-\lambda)y_2$. Choosing $\alpha \in (0,1)$ small enough so that $0 < \lambda - \alpha < \lambda + \alpha < 1$, we can write $x = \frac{1}{2}(z_1 + z_2)$ with $z_1 = (\lambda-\alpha)y_1 + (1-\lambda+\alpha)y_2 \in B$ and $z_2 = (\lambda+\alpha)y_1 + (1-\lambda-\alpha)y_2 \in B$. By the triangle inequality, we obtain

$$1 = \|x\| = \left\|\frac{1}{2}(z_1 + z_2)\right\| \leq \frac{1}{2}\|z_1\| + \frac{1}{2}\|z_2\|,$$

which can be satisfied for $\max\{\|z_1\|, \|z_2\|\} \leq 1$ if and only if $\|z_1\| = 1 = \|z_2\|$. Strict convexity now ensures that $z_1 = z_2 = x$, which, in turn, yields $y_1 = y_2 = x$. Hence, $x \in \mathrm{ex}(B)$. As $x \in S$ was arbitrary, we showed $S \subseteq \mathrm{ex}(B)$, as desired.

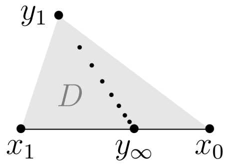
"$\Leftarrow$": Suppose that $S = \mathrm{ex}(B)$. Hence, for all $x, y \in X$ with $\|x\| = \|y\| = \|\frac{x+y}{2}\| = 1$ we get $x = y$. This shows that X is strictly convex.

8.2. Closedness/Non-closedness of sets of extremal points

(a) Let $K \subseteq \mathbb{R}^2$ be a closed convex subset. Prove that the set E of all extremal points of K is closed.

Solution: It is clear that the set E of extremal points of the closed convex subset $K \subseteq \mathbb{R}^2$ must be a subset of the boundary ∂K of K because the center of every ball contained in K is a convex combination of other points in this ball.

Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in E which converges to some $y_\infty \in K$. Suppose $y_\infty \notin E$. Then there exist distinct points $x_1, x_0 \in K$ and some $0 < \lambda < 1$ such that $\lambda x_1 + (1-\lambda)x_0 = y_\infty$. For any $n \in \mathbb{N}$, the point y_n is extremal and therefore cannot lie on the segment between x_1 and x_0. Intuitively, the sequence $(y_n)_{n\in\mathbb{N}}$ must approach y_∞ from "above" or "below" this segment. By restriction to a subsequence, we can assume that all y_n strictly lie on the same side of the the affine line through x_1 and x_2. By convexity of K, the triangle $D = \mathrm{conv}\{x_1, x_0, y_1\}$ is a subset of K. The arguments above and convergence $y_n \to y_\infty$ imply that for $n \in \mathbb{N}$ sufficiently large, y_n is in the interior of D and thus in the interior of K. This however contradicts $(y_n)_{n\in\mathbb{N}} \subseteq E \subseteq \partial K$. We conclude $y \in E$ which proves that E is closed.



(b) Consider the convex hull C of the circle $\{(1+\cos(\varphi), \sin(\varphi), 0)\colon 0 \le \varphi \le 2\pi\}$ and the points $(0,0,\pm 1)$ in $\mathbb{R}^3$. Determine the extremal points of C.

Solution: The convex hull is a circular double cone with the two tips joined by a straight line segment on the z-axis. Thus the extremal points are the given points (circle and $(0,0,\pm 1)$) minus the origin.

8.3. Birkhoff–von Neumann theorem

A matrix $M = (M_{ij})_{1\le i,j\le n} \in \mathbb{R}^{n\times n}$ (where $n \in \mathbb{N}$) with $M_{ij} \ge 0$ is called *doubly stochastic* iff its rows and columns all add up to one: $\sum_{i=1}^n M_{ij} = 1 = \sum_{i=1}^n M_{ji}$. Prove that every doubly stochastic matrix is a convex combination of permutation matrices.

Hint: Suppose M is a doubly stochastic matrix. Find a permutation matrix P and $\lambda \in (0,\infty)$ such that $N = M - \lambda P$ has non-negative entries. If $N \ge 0$ then $\frac{1}{1-\lambda} N$ is doubly stochastic. One way to find P is as follows:

Recall **Hall's Marriage Theorem**: Assume X and Y are finite sets and let $\Gamma \subseteq X \times Y$. The following statements are equivalent:

(i) There exists an injective function $f\colon X \to Y$ whose graph is contained in Γ.

(ii) For every $A \subseteq X$ the set $\Gamma(A) = \{y \in Y \mid (x,y) \in \Gamma \text{ for some } x \in A\}$ satisfies $\#\Gamma(A) \ge \#A$.

Let $X = Y = \{1,2,\dots,n\}$ and let $\Gamma = \{(i,j) \in X \times Y \mid M_{ij} > 0\}$. Use the fact that M is doubly stochastic to verify condition (ii). The injective map $f\colon X \to Y$ from (i) determines a permutation of $\{1,2,\dots,n\}$.

Solution: We verify the condition of Hall's marriage theorem for the set Γ proposed in the above hint. Consider now an arbitrary set $A \subseteq X$. Since the rows and columns of M all sum to one, we have:

$$\#\Gamma(A) = \sum_{(i,j)\in X\times\Gamma(A)} M_{ij} \ge \sum_{(i,j)\in A\times\Gamma(A)} M_{ij} \ge \sum_{(i,j)\in (A\times Y)\cap\Gamma} M_{ij} = \sum_{(i,j)\in A\times Y} M_{ij} = \#A.$$

Now Hall's marriage theorem provides us with an injective map $f\colon X \to Y$ whose graph is contained in Γ. Note that f is bijective because $\#X = \#Y$. The permutation matrix $P = (P_{ij})_{1\le i,j\le n} \in \mathbb{R}^{n\times n}$ associated with f is given by

$$P_{ij} = \begin{cases} 1 & \text{if } j = f(i) \\ 0 & \text{otherwise.} \end{cases}$$

Since the graph of f is contained in Γ, we have $\lambda = \min_{1\le i\le n} M_{i,f(i)} > 0$. Clearly, $N := M - \lambda P$ has non-negative entries. If $N = 0$ then $\lambda = 1$, and M is a permutation matrix, otherwise $\lambda < 1$ and $M = \lambda P + (1-\lambda)\frac{1}{1-\lambda}N$ is a genuine convex combination of two doubly stochastic matrices, namely P and $\frac{1}{1-\lambda}N$. Thus, we obtained that permutation matrices are the extremal points in the convex set of doubly stochastic matrices. Birkhoff–von Neumann's theorem now follows from Krein–Milman's theorem.

8.4. Topologies induced by linear functionals

Let X be a real vector space.

(a) Let $n \in \mathbb{N}$ and let $\varphi_1, \varphi_2, \ldots, \varphi_n, \psi\colon X \to \mathbb{R}$ be linear functionals. Prove that the following are equivalent:

(i) There exist $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ satisfying $\psi = \sum_{k=1}^n \lambda_k \varphi_k$.

(ii) There is a constant $C \in (0,\infty)$ such that $|\psi(x)| \le C \max_{1\le k\le n}|\varphi_k(x)|$ for all $x \in X$.

(iii) $\ker(\psi) \supseteq \bigcap_{k=1}^n \ker(\varphi_k)$.

Solution: "*(i) ⇒ (ii)*": With $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ such that $\psi = \sum_{k=1}^n \lambda_k \varphi_k$, we obtain for all $x \in X$ that

$$|\psi(x)| \le \sum_{k=1}^n |\lambda_k||\varphi_k(x)| \le \max_{1\le k\le n}|\varphi_k(x)| \sum_{k=1}^n |\lambda_k|.$$

That is, (ii) holds with $C = \sum_{k=1}^n |\lambda_k|$ if the sum is not 0, in which case any $C \in (0,\infty)$ works.

"*(ii) ⇒ (iii)*": With $C \in (0,\infty)$ such that $|\psi(x)| \le C \max_{1\le k\le n}|\varphi_k(x)|$ for all $x \in X$, we clearly obtain for every $x \in \bigcap_{k=1}^n \ker(\varphi_k)$ that $\psi(x) = 0$. That is, (iii) holds.

"*(iii) ⇒ (i)*": Consider the linear function $\phi\colon X \to \mathbb{R}^n$ given by

$$\phi(x) = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x)) \quad \text{for all } x \in X.$$

Note that for all $x, y \in X$ it holds that $x - y \in \ker(\phi)$ if and only if $\varphi_k(x) = \varphi_k(y)$ for all $k \in \{1, 2, \dots, n\}$. Thus, $X/\ker(\phi)$ is isomorphic to $\operatorname{im}(\phi)$. Moreover, since $\ker(\psi) \supseteq \bigcap_{k=1}^{n} \ker(\varphi_k) = \ker(\phi)$, we see that there is a well-defined linear map $l\colon \operatorname{im}(\phi) \to \mathbb{R}$ satisfying

$$l(\phi(x)) = \psi(x) \quad \text{for all } x \in X.$$

We know from linear algebra (if you insist you can also invoke the Hahn–Banach theorem) that there exists a linear extension $L\colon \mathbb{R}^n \to \mathbb{R}$ of l. Moreover do we know from linear algebra (if you insist you can also invoke Riesz's representation theorem for Hilbert spaces) that there exist $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$ such that $L(y) = \sum_{k=1}^{n} \lambda_k y_k$ for all $y = (y_1, y_2 \dots, y_n) \in \mathbb{R}^n$. This ensures in particular that

$$\psi(x) = l(\phi(x)) = L(\phi(x)) = \sum_{k=1}^{n} \lambda_k \varphi_k(x) \quad \text{for all } x \in X.$$

(b) Let $F \subseteq \{f \mid X \to \mathbb{R}\colon f \text{ is linear}\}$ be a family of linear functionals and let $\mathcal{U}_F$ be the topology on X induced by F. Prove that

$$\operatorname{span}(F) = \{\varphi\colon X \to \mathbb{R} \mid \varphi \text{ is } \mathcal{U}_F\text{-continuous and linear}\}.$$

Solution: If $\varphi\colon X \to \mathbb{R}$ is linear and $\mathcal{U}_F$-continuous, then the set $\varphi^{-1}((-1,1))$ is $\mathcal{U}_F$-open. Hence, there are $f_1, \dots, f_n \in F$ and $\varepsilon \in (0, \infty)$ such that

$$\varphi^{-1}((-1,1)) \supseteq \bigcap_{k=1}^{n} f_k^{-1}((-\varepsilon, \varepsilon)).$$

By linearity, we infer for every $m \in \mathbb{N}$ that

$$\varphi^{-1}((-\tfrac{1}{m}, \tfrac{1}{m})) \supseteq \bigcap_{k=1}^{n} f_k^{-1}((-\tfrac{\varepsilon}{m}, \tfrac{\varepsilon}{m})). \tag{1}$$

Letting $m \to \infty$, we obtain that $\varphi(x) = 0$ for all $x \in \bigcap_{k=1}^{m} \ker(f_k)$. Part (a) above now ensures that $\varphi \in \operatorname{span}(\{f_k \mid k \in \{1, 2, \dots, n\}\}) \subseteq \operatorname{span}(F)$.

Remark: We checked condition (i) in (a). We could also have deduced from (1) that $|\varphi(x)| \le \frac{1}{\varepsilon} \max_{1 \le k \le n} |f_k(x)|$ for every $x \in X$, in other words: condition (ii) of (a).

(c) Suppose X is a normed space. Consider a weak*-continuous linear functional $\varphi\colon X^* \to \mathbb{R}$. Prove that there is $x \in X$ such that $\varphi(f) = f(x)$ for all $f \in X^*$.

Solution: This follows immediately from (b) when taking $F = \{X^* \ni \varphi \mapsto \varphi(x) \in \mathbb{R}\colon x \in X\}$ and noticing that $F = \operatorname{span}(F)$.

8.5. Weak topologies are in general non-metrizable

Let $(X, \|\cdot\|_X)$ be a normed space and let τ_{w} denote the weak topology on X. This exercise's goal is to show that τ_{w} is not metrizable if X is infinite-dimensional. Let us start by recalling what a *neighbourhood basis* is and what it means for a topology to be *metrizable*:

- (*Neighbourhood basis*) Let (Y, τ) be a topological space. Denoting the set of all neighbourhoods of a point $y \in Y$ by
$$\mathcal{U}_y = \{U \subseteq Y \mid \exists\, O \in \tau : \ y \in O \subseteq U\},$$
we call $\mathcal{B}_y \subseteq \mathcal{U}_y$ a *neighbourhood basis* of y in (Y, τ), if $\forall\, U \in \mathcal{U}_y \, \exists V \in \mathcal{B}_y \colon V \subseteq U$.

- (*Metrizability*) A topological space (Y, τ) is called *metrizable* if there exists a metric $d\colon Y \times Y \to \mathbb{R}$ on Y denoting $B_\varepsilon(a) = \{y \in Y \mid d(y, a) < \varepsilon\}$ (for $a \in Y$, $\varepsilon \in (0, \infty)$), there holds
$$\tau = \{O \subseteq Y \mid \forall a \in O \ \exists\, \varepsilon > 0 : \ B_\varepsilon(a) \subseteq O)\}.$$

(a) Show that any metrizable topology τ satisfies the *first axiom of countability* which means that each point has a *countable* neighbourhood basis.

Solution: Let (Y, τ) be a metrizable topological space. Let $d\colon Y \times Y \to \mathbb{R}$ be a metric inducing the topology τ. Given $y \in Y$, we consider

$$B_\varepsilon(y) := \{z \in Y \mid d(y, z) < \varepsilon\} \text{ for } \varepsilon \in (0, \infty), \qquad \mathcal{B}_y := \left\{B_{\frac{1}{n}}(y) \mid n \in \mathbb{N}\right\}.$$

Let U now be any neighbourhood of y. Since (Y, τ) is metrizable, there exists $\varepsilon \in (0, \infty)$ such that $B_\varepsilon(y) \subseteq U$. Choosing $\mathbb{N} \ni n > \frac{1}{\varepsilon}$, we have $B_{\frac{1}{n}}(y) \subseteq U$, which shows that $\mathcal{B}_y$ is a neighbourhood basis of y in (Y, τ). Since $y \in Y$ is arbitrary and $\mathcal{B}_y$ countable, we have verified the first axiom of countability for (Y, τ).

(b) Prove that

$$\mathcal{B} := \left\{ \bigcap_{k=1}^{n} f_k^{-1}\Big((-\varepsilon, \varepsilon)\Big) \ \Big| \ n \in \mathbb{N}, \ f_1, \ldots, f_n \in X^*, \ \varepsilon > 0 \right\}$$

is a neighbourhood basis of $0 \in X$ in (X, τ_{w}).

Solution: Let $U \subseteq X$ be any neighbourhood of $0 \in X$ in (X, τ_{w}). Then there exists $\Omega \in \tau_{\mathrm{w}}$ such that $0 \in \Omega \subseteq U$. By definition of weak topology, Ω is an arbitrary union and finite intersection of sets of the form $f^{-1}(I)$ for $f \in X^*$ and $I \subseteq \mathbb{R}$ open. In particular, Ω contains a finite intersection of such sets containing the origin. More precisely, there exist $f_1, \ldots, f_n \in X^*$ and open sets $I_1, \ldots, I_n \subseteq \mathbb{R}$ such that

$$\Omega \supseteq \bigcap_{k=1}^{n} f_k^{-1}(I_k) \ni 0.$$

By linearity $f_k(0) = 0 \in I_k$ for every $k \in \{1, \ldots, n\}$. Since $I_1, \ldots, I_n \subseteq \mathbb{R}$ are open and n finite, there exists $\varepsilon \in (0, \infty)$ such that $(-\varepsilon, \varepsilon) \subseteq I_k$ for every $k \in \{1, \ldots, n\}$. Thus,

$$\Omega \supseteq \bigcap_{k=1}^{n} f_k^{-1}\Big((-\varepsilon, \varepsilon)\Big) = \{x \in X \mid \forall k \in \{1, \ldots, n\} \colon |f_k(x)| < \varepsilon\}$$

and we conclude that a neighbourhood basis of $0 \in X$ in (X, τ_{w}) is given by

$$\mathcal{B} := \left\{ \bigcap_{k=1}^{n} f_k^{-1}\Big((-\varepsilon, \varepsilon)\Big) \;\middle|\; n \in \mathbb{N},\ f_1, \ldots, f_n \in X^*,\ \varepsilon \in (0, \infty) \right\}.$$

(c) Show that if (X, τ_{w}) is first countable, then $(X^*, \|\cdot\|_{X^*})$ admits a countable algebraic basis.

Solution: Let $(X, \|\cdot\|_X)$ be a normed space and suppose that (X, τ_{w}) is first countable. Then there exists a countable neighbourhood basis $\{A_\alpha\}_{\alpha \in \mathbb{N}}$ of $0 \in X$ in (X, τ_{w}). Since $\mathcal{B}$ defined in (b) is also a neighbourhood basis of $0 \in X$ in (X, τ_{w}), we have

$$\forall \alpha \in \mathbb{N} \quad \exists B_\alpha \in \mathcal{B} : \quad B_\alpha \subseteq A_\alpha.$$

By construction of $\mathcal{B}$, this means that

$$\forall \alpha \in \mathbb{N} \quad \exists n_\alpha \in \mathbb{N},\ f_1^\alpha, \ldots, f_{n_\alpha}^\alpha \in X^*,\ \varepsilon_\alpha \in (0, \infty) :$$
$$B_\alpha := \{x \in X \mid \forall k \in \{1, \ldots, n_\alpha\} \colon |f_k^\alpha(x)| < \varepsilon_\alpha\} \subseteq A_\alpha.$$

In other words, the topology τ_{w} coincides with the topology $\mathcal{U}_F$ which is induced by $F = \bigcup_{\alpha \in \mathbb{N}} \bigcup_{k=1}^{n_\alpha} \{f_k^\alpha\}$ (cf. Problem 8.4 (*Topologies induced by linear functionals*)). According to 8.4(b), $X^* \subseteq \operatorname{span}(F)$. In other words, F contains an algebraic basis of X^* and F is clearly countable.

(d) Assume that X is infinite-dimensional and conclude from (a), (c) and Problem 2.2 (*Algebraic bases for Banach spaces*) that (X, τ_{w}) is not metrizable.

Solution: By (a) and (c), $(X^*, \|\cdot\|_{X^*})$ admits a countable algebraic basis. But since X is infinite-dimensional, $(X^*, \|\cdot\|_{X^*})$ is infinite-dimensional. Moreover, $(X^*, \|\cdot\|_{X^*})$ is a Banach space. But as such, according to Problem 2.2, it can only have a countable algebraic basis if it is finite-dimensional, a contradiction.

8.6. Weak and weak* topology on ℓ^1

Let $e_n = (\delta_{kn})_{k\in\mathbb{N}} \subseteq \mathbb{R}$ for every $n \in \mathbb{N}$. For $p \in (1,\infty)$, $(e_n)_{n\in\mathbb{N}} \subseteq \ell^p$ converges to 0 with respect to both weak as well as weak* convergence in ℓ^p as $n \to \infty$. ℓ^1 behaves similarly with respect to weak* convergence, but differently with respect to weak convergence:

(a) Show that $(e_n)_{n\in\mathbb{N}} \subseteq \ell^1$ does not converge weakly to 0 in ℓ^1.

Solution: Let $\varphi \in (\ell^1)^*$ be given by $\varphi(x) = \sum_{n=1}^{\infty} x_n$ for $x = (x_n)_{n\in\mathbb{N}} \in \ell^1$ (in other words, φ is the element of the dual space of ℓ^1 which is represented by the constant sequence $(1)_{n\in\mathbb{N}} \in \ell^\infty$). Then we obtain that $\varphi(e_n) = 1$ for all $n \in \mathbb{N}$, contradicting $e_n \xrightarrow{w} 0$ in ℓ^1.

(b) Viewing ℓ^1 as the dual space of c_0 (cf. Problem 7.2 (*Dual spaces of c_0 and c*)), argue that $(e_n)_{n\in\mathbb{N}}$ converges to zero in the weak* topology.

Solution: We identify ℓ^1 with $(c_0)^*$ via the mapping $\Phi = (\ell^1 \ni (x_n)_{n\in\mathbb{N}} \mapsto (c_0 \ni (y_n)_{n\in\mathbb{N}} \mapsto \sum_{n=1}^{\infty} x_n y_n \in \mathbb{R}) \in (c_0)^*)$. With this, we obtain for every $y = (y_n)_{n\in\mathbb{N}} \in c_0$ that

$$0 = \lim_{n\to\infty} y_n = \lim_{n\to\infty} [\Phi(e_n)](y),$$

that is, $(\Phi(e_n))_{n\in\mathbb{N}} \xrightarrow{w^*} 0$ in $(c_0)^*$ as $n \to \infty$. And this is exactly what we meant by saying that $e_n \xrightarrow{w^*} 0$ in ℓ^1 with ℓ^1 being viewed as $(c_0)^*$.

(c) (*Schur's Theorem.*) Let $(x_n)_{n\in\mathbb{N}} \subseteq \ell^1$ be converging weakly to 0. Prove that $\|x_n\|_{\ell^1} \to 0$ as $k \to \infty$.

Solution: Suppose that $\|x_n\|_{\ell^1}$ does not converge to zero as $n \to \infty$. After passing to a subsequence there is $\eta \in (0,\infty)$ such that $\|x_n\|_{\ell^1} \geq \eta$ for all $n \in \mathbb{N}$. Note that for every $K \in \mathbb{N}$ it holds that $f_K\colon \ell^1 \to \mathbb{R}$, defined by $f_K(y) = \sum_{k=1}^{K} |y_k|$ for all $y = (y_k)_{k\in\mathbb{N}} \in \ell^1$, is weakly continuous. Hence, we obtain for every $K \in \mathbb{N}$ that $f_K(x_n) \to 0$ as $n \to \infty$. Thus, there exists $J\colon \mathbb{N} \times (0,\infty) \to \mathbb{N}$ satisfying that

$$\sup_{j \geq J(K,\varepsilon)} f_K(x_j) \leq \varepsilon \quad \text{for all } K \in \mathbb{N}, \varepsilon \in (0,\infty).$$

Moreover, there exists $L\colon \mathbb{N} \times (0,\infty) \to \mathbb{N}$ satisfying that (using the notation $x_n = (x_{n,j})_{j\in\mathbb{N}}$)

$$\sum_{j=1}^{L(n,\varepsilon)} |x_{n,j}| \geq \|x_n\|_{\ell^1} - \varepsilon \quad \text{for all } n \in \mathbb{N}, \varepsilon \in (0,\infty).$$

Now, define $(K_j)_{j\in\mathbb{N}_0} \subseteq \mathbb{N}$, $(n_j)_{j\in\mathbb{N}_0} \subseteq \mathbb{N}$ so that

- $K_0 = n_0 = 1$,
- $n_j = \max\{J(K_{j-1}, \frac{1}{j}), n_{j-1} + 1\}$ for all $j \in \mathbb{N}$,
- $K_j = \max\{L(n_j, \frac{1}{j}), K_{j-1} + 1\}$ for all $j \in \mathbb{N}$.

Note that $(n_j)_{j\in\mathbb{N}_0}$ and $(K_j)_{j\in\mathbb{N}_0}$ are strictly increasing. In addition, the fact that for all $j \in \mathbb{N}$ it holds that $n_j \geq J(K_{j-1}, \frac{1}{j})$ as well as $K_j \geq L(n_j, \frac{1}{j})$ implies for all $j \in \mathbb{N}$ that

$$\sum_{k=1}^{K_{j-1}} |x_{n_j,k}| \leq \frac{1}{j} \quad \text{and} \quad \sum_{k=1}^{K_j} |x_{n_j,k}| \geq \|x_{n_j}\|_{\ell^1} - \frac{1}{j}.$$

In particular, it holds for all $j \in \mathbb{N}$ that

$$\sum_{k=K_{j-1}+1}^{K_j} |x_{n_j,k}| \geq \|x_{n_j}\|_{\ell^1} - \frac{2}{j} \quad \text{and} \quad \sum_{k=K_j+1}^{\infty} |x_{n_j,k}| \leq \frac{1}{j}.$$

Hence, for $y = (y_k)_{k\in\mathbb{N}} \in \ell^\infty$, defined via

$$y_k = \begin{cases} 0 & : k = 1, \\ \operatorname{sign}(x_{n_j,k}) & : \text{if } K_{j-1} < k \leq K_j \text{ for some } j \in \mathbb{N}, \end{cases}$$

we obtain, by what was deduced above, for every $j \in \mathbb{N}$ that

$$\begin{aligned} \sum_{k=1}^{\infty} y_k x_{n_j,k} &= \sum_{k=1}^{K_{j-1}} y_k x_{n_j,k} + \sum_{k=K_{j-1}+1}^{K_j} y_k x_{n_j,k} + \sum_{k=K_j+1}^{\infty} y_k x_{n_j,k} \\ &\geq -\sum_{k=1}^{K_{j-1}} |x_{n_j,k}| + \sum_{k=K_{j-1}+1}^{K_j} |x_{n_j,k}| - \sum_{k=K_j+1}^{\infty} |x_{n_j,k}| \\ &\geq \|x_{n_j}\|_{\ell^1} - \frac{4}{j} \geq \eta - \frac{4}{j}. \end{aligned}$$

This implies that $\limsup_{n\to\infty} \sum_{k=1}^{\infty} y_k x_{n,k} \geq \eta > 0$, contradicting $x_n \xrightarrow{w} 0$ in ℓ^1.