10.1. Various notions of continuity

Suppose $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed $\mathbb{K}$-vector spaces (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$).

(a) A linear map $A\colon X \to Y$ is bounded if and only if it is $\sigma(X, X^*)$-$\sigma(Y, Y^*)$-continuous (i.e., continuous with respect to the weak topologies on X and Y).

Solution: "($\Rightarrow$)": Assume that A is bounded. Let $O \in \sigma(Y, Y^*)$ be arbitrary but fixed. We need to show that $A^{-1}(O) \in \sigma(X, X^*)$. For this, let $x \in A^{-1}(O)$ be arbitrary but fixed. Then it holds that $Ax \in O$ and – by $O \in \sigma(Y, Y^*)$ – there exist $\varepsilon \in (0, \infty)$, $n \in \mathbb{N}$, $y_1^*, y_2^*, \ldots, y_n^* \in Y^*$ satisfying that

$$\{y \in Y \mid |y_i^*(y - Ax)| < \varepsilon \text{ for all } i \in \{1, 2, \ldots, n\}\} \subseteq O.$$

Since $A \in L(X, Y)$, it holds for all $i \in \{1, 2, \ldots, n\}$ that $y_i^* \circ A \in X^*$ and therefore

$$\begin{aligned}&\{\xi \in X \mid |(y_i^* \circ A)(\xi - x)| < \varepsilon \text{ for all } i \in \{1, 2, \ldots, n\}\}\\ &= A^{-1}(\{y \in Y \mid |y_i^*(y - Ax)| < \varepsilon \text{ for all } i \in \{1, 2, \ldots, n\}\}) \subseteq A^{-1}(O).\end{aligned}$$

As $x \in A^{-1}(O)$ was arbitrary, this shows that $A^{-1}(O)$ is open. As $O \in \sigma(Y, Y^*)$ was arbitrary, we proved that A is $\sigma(X, X^*)$-$\sigma(Y, Y^*)$-continuous.

"($\Leftarrow$)": Assuming that A is $\sigma(X, X^*)$-$\sigma(Y, Y^*)$-continuous, we obtain for every $y^* \in Y^*$ that $y^* \circ A$ is $\sigma(X, X^*)$-continuous. From part (b) in Problem 8.4 (*Topologies induced by linear functionals*), we know that a linear functional on X is $\sigma(X, X^*)$-continuous if and only if it belongs to X^*. Thus, we obtain that $y^* \circ A \in X^*$ for every $y^* \in Y^*$. In particular, it holds for every $y^* \in Y^*$ that there exists $C \in [0, \infty)$ satisfying $\forall x \in X\colon |y^*(Ax)| \leq C\|x\|_X$. On the other hand, it clearly holds for every $x \in X$ that $\forall y^* \in Y^*\colon |y^*(Ax)| \leq \|Ax\|_Y \|y^*\|_{Y^*}$. Hence, the bilinear mapping $Y^* \times X \ni (y^*, x) \mapsto y^*(Ax) \in \mathbb{K}$ satisfies the conditions of part (b) in Problem 5.3 (*Continuity of bilinear maps*), which in turn guarantees that there exists $C \in [0, \infty)$ such that

$$|y^*(Ax)| \leq C\|x\|_X \|y^*\|_{Y^*} \quad \text{for all } x \in X, y^* \in Y^*.$$

This implies that $\|Ax\|_Y \leq C\|x\|_X$ for all $x \in X$. (*Alternatively*, redo the proof of Problem 5.3 in this special case: since $(Y^*, \|\cdot\|_{Y^*})$ is complete and since for all $y^* \in Y^*$ it holds that $\sup_{x \in X, \|x\|_X \leq 1} |y^*(Ax)| = \sup_{x \in X, \|x\|_X \leq 1} |(y^* \circ A)(x)| = \|y^* \circ A\|_{X^*} < \infty$, the Banach–Steinhaus theorem implies that $\sup_{x \in X, \|x\|_X \leq 1} \|Y^* \ni y^* \mapsto y^*(Ax) \in \mathbb{K}\|_{Y^{**}} < \infty$.)

(b) A linear map $B\colon Y^* \to X^*$ is $\sigma(Y^*, Y)$-$\sigma(X^*, X)$-continuous (i.e., continuous with respect to the weak* topologies on Y^* and X^*) if and only if there is a bounded linear operator $A\colon X \to Y$ such that $B = A^*$.

"(⇐)": Assume that $B = A^*$ for some $A \in L(X,Y)$. This implies that $B \in L(Y^*, X^*)$. From now on, the proof is analogous to the corresponding part of the proof of (a). Let $O \in \sigma(X^*, X)$ be arbitrary but fixed. We need to show that $B^{-1}(O) \in \sigma(Y^*, Y)$. For this, let $y^* \in B^{-1}(O)$ be arbitrary but fixed. Then it holds that $By^* \in O$ and – by $O \in \sigma(X^*, X)$ – there exist $\varepsilon \in (0,\infty)$, $n \in \mathbb{N}$, $x_1, x_2, \ldots, x_n \in X$ satisfying that

$$\{x^* \in X^* \mid |(By^* - x^*)(x_i)| < \varepsilon \text{ for all } i \in \{1,2,\ldots,n\}\} \subseteq O.$$

Since $B = A^*$ and $A \in L(X,Y)$, we obtain that

$$\begin{aligned}&\{v^* \in Y^* \mid |(y^* - v^*)(Ax_i)| < \varepsilon \text{ for all } i \in \{1,2,\ldots,n\}\}\\ &= \{v^* \in Y^* \mid |(By^* - Bv^*)(x_i)| < \varepsilon \text{ for all } i \in \{1,2,\ldots,n\}\}\\ &= B^{-1}(\{x^* \in X^* \mid |(By^* - x^*)(x_i)| < \varepsilon \text{ for all } i \in \{1,2,\ldots,n\}\}) \subseteq B^{-1}(O).\end{aligned}$$

As $y^* \in B^{-1}(O)$ was arbitrary, this shows that $B^{-1}(O)$ is open. As $O \in \sigma(X^*, X)$ was arbitrary, we proved that B is $\sigma(Y^*,Y)$-$\sigma(X^*,X)$-continuous.

"(⇒)": Assuming that B is $\sigma(Y^*,Y)$-$\sigma(X^*,X)$-continuous, it holds for every $x \in X$ that $Y^* \ni y^* \mapsto (By^*)(x) \in \mathbb{K}$ is $\sigma(Y^*,Y)$-continuous. Problem 8.4 (*Topologies induced by linear functionals*) again assures that for every $x \in X$ there exists a unique element of Y, called Ax from now on, such that

$$(By^*)(x) = y^*(Ax) \quad \text{for all } x \in X, y^* \in Y^*.$$

Clearly, $X \ni x \mapsto Ax \in Y$ is linear (by uniqueness of Ax for $x \in X$ and linearity of everything else). This and the above relation show that $B = A^*$. It remains to show that A is bounded. For this, we note that

- for every $y^* \in Y^*$, it holds that

$$|y^*(Ax)| = |(By^*)(x)| \leq \|By^*\|_{X^*}\|x\|_X$$

and

- for every $x \in X$, it holds that

$$|y^*(Ax)| \leq \|Ax\|_Y\|y^*\|_{Y^*}.$$

Part (b) in Problem 5.3 (*Continuity of bilinear maps*) ensures again that there exists $C \in [0,\infty)$ such that $|y^*(Ax)| \leq C\|x\|_X\|y^*\|_{Y^*}$ for all $x \in X$, $y^* \in Y^*$.

(c) A linear operator $A\colon X \to Y$ is $\sigma(X,X^*)$-$\|\cdot\|_Y$-continuous (i.e., weak–norm continuous) if and only if it is bounded and has finite rank (i.e., has finite-dimensional range).

Solution: "($\Rightarrow$)": By definition, there exist $\varepsilon \in (0,\infty)$, $n \in \mathbb{N}$ and $x_1^*, x_2^*, \ldots, x_n^* \in X^*$ such that for all $x \in X$ satisfying $|x_i^*(x)| < \varepsilon$ it holds that $\|Ax\|_Y < 1$. This implies that

$$\|Ax\|_Y \leq \frac{1}{\varepsilon} \max_{1 \leq k \leq n} |x_k^*(x)| \quad \text{for all} \quad x \in X. \tag{1}$$

In particular, it holds for all $x, y \in X$ with $x_k^*(x) = x_k^*(y)$ for all $k \in \{1, 2, \ldots, n\}$ that $Ax = Ay$. Hence, the mapping

$$\{(x_1^*(x), x_2^*(x), \ldots, x_n^*(x)) \mid x \in X\} \ni (x_1^*(x), x_2^*(x), \ldots, x_n^*(x)) \mapsto Ax \in Y$$

is well-defined and linear. Moreover, with the domain space being finite-dimensional, this map can only have finite-dimensional image. The image, though, is $A(X)$. Hence, A itself has finite-dimensional image. Boundedness of A is clear from (1) (or, as one could say, was clear from the beginning, since A is also $\sigma(X, X^*)$-$\sigma(Y, Y^*)$-continuous).

"($\Leftarrow$)": Assume that $A \in L(X, Y)$ has finite rank. Let $y_1, y_2, \ldots, y_n \in Y$ be a basis of $\operatorname{im}(A)$ and let $y_1^*, y_2^*, \ldots, y_n^* \in Y^*$ be a dual basis, i.e. $y_i^*(y_j) = \delta_{ij}$ for all $i, j \in \{1, 2, \ldots, n\}$. This ensures that

$$Ax = \sum_{i=1}^{n} y_i^*(Ax) y_i \quad \text{for all } x \in X.$$

Let $O \subseteq Y$ be an arbitrary but fixed open set w.r.t. the norm topology. We want to show that $A^{-1}(O) \subseteq X$ is open. For this, let $x \in A^{-1}(O)$ be arbitrary but fixed. Then $Ax \in O$ and – as O is open w.r.t. the norm topology – there exists $\varepsilon \in (0, \infty)$ such that $\{y \in Y \mid \|y - Ax\|_Y < \varepsilon\} \subseteq O$. Now, since $y_i^* \in Y^*$ for every $i \in \{1, 2, \ldots, n\}$ and $A \in L(X, Y)$ we have that $y_i^* \circ A \in X^*$ for every $i \in \{1, 2, \ldots, n\}$ and, therefore, the first set below is a $\sigma(X, X^*)$-neighborhood of x:

$$\begin{aligned}
&\left\{\xi \in X \mid |(y_i^* \circ A)(\xi - x)| < \frac{\varepsilon}{n\|y_i\|_Y} \text{ for all } i \in \{1, 2, \ldots, n\}\right\} \\
&\subseteq \left\{\xi \in X \mid \sum_{i=1}^{n} |y_i^*(A\xi - Ax)| \|y_i\|_Y < \varepsilon\right\} \\
&\subseteq \{\xi \in X \mid \|A\xi - Ax\|_Y < \varepsilon\} \subseteq A^{-1}(O).
\end{aligned}$$

As $x \in A^{-1}(O)$ was arbitrary, we obtain that $A^{-1}(O) \in \sigma(X, X^*)$. As $O \subseteq Y$ was an arbitrary open set w.r.t. the norm topology, we have that A is $\sigma(X, X^*)$-$\|\cdot\|_Y$-continuous.

10.2. Elementary properties of dual operators

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed $\mathbb{K}$-vector spaces (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$). Recall that if $T \in L(X, Y)$, then its dual operator T^* is in $L(Y^*, X^*)$ and it is characterised by the property

$$\langle T^* y^*, x\rangle_{X^* \times X} = \langle y^*, Tx\rangle_{Y^* \times Y} \quad \text{for every } x \in X \text{ and } y^* \in Y^*.$$

Prove the following facts about dual operators.

(a) $(\mathrm{Id}_X)^* = \mathrm{Id}_{X^*}$.

Solution: Let $x \in X$ and $x^* \in X^*$ be arbitrary. By definition of $(\mathrm{Id}_X)^* \colon X^* \to X^*$, we have

$$\Big\langle (\mathrm{Id}_X)^* x^*, x\Big\rangle_{X^* \times X} = \Big\langle x^*, \mathrm{Id}_X\, x\Big\rangle_{X^* \times X} = \langle x^*, x\rangle_{X^* \times X^*}.$$

Since $x \in X$ is arbitrary, $(\mathrm{Id}_X)^* x^* = x^*$. Since $x^* \in X^*$ is arbitrary, $(\mathrm{Id}_X)^* = \mathrm{Id}_{(X^*)}$.

(b) If $T \in L(X, Y)$ and $S \in L(Y, Z)$, then $(S \circ T)^* = T^* \circ S^*$.

Solution: Let $z^* \in Z^*$ and $x \in X$ be arbitrary. Then, $(S \circ T)^* = T^* \circ S^*$ follows from

$$\begin{aligned}\Big\langle (S \circ T)^* z^*, x\Big\rangle_{X^* \times X} &= \Big\langle z^*, S(Tx)\Big\rangle_{Z^* \times Z} \\ &= \Big\langle S^* z^*, Tx\Big\rangle_{Y^* \times Y} = \Big\langle T^*(S^* z^*), x\Big\rangle_{X^* \times X}.\end{aligned}$$

(c) If $T \in L(X, Y)$ is bijective with inverse $T^{-1} \in L(Y, X)$, then $(T^*)^{-1} = (T^{-1})^*$.

Solution: To prove $(T^*)^{-1} = (T^{-1})^*$, we apply the results from (a) and (b) and obtain

$$\begin{aligned}&T^* \circ (T^{-1})^* = (T^{-1} \circ T)^* = (\mathrm{Id}_X)^* = \mathrm{Id}_{X^*}, \\ &(T^{-1})^* \circ T^* = (T \circ T^{-1})^* = (\mathrm{Id}_Y)^* = \mathrm{Id}_{Y^*}.\end{aligned}$$

(d) Let $\mathcal{I}_X \colon X \hookrightarrow X^{**}$ and $\mathcal{I}_Y \colon Y \hookrightarrow Y^{**}$ be the canonical inclusions. Then,

$$\forall\, T \in L(X, Y): \quad \mathcal{I}_Y \circ T = (T^*)^* \circ \mathcal{I}_X.$$

Solution: Let $x \in X$ and $y^* \in Y^*$ be arbitrary. Then, $(\mathcal{I}_Y \circ T) = (T^*)^* \circ \mathcal{I}_X$ follows from

$$\begin{aligned}\Big\langle (\mathcal{I}_Y \circ T)x, y^*\Big\rangle_{Y^{**} \times Y^*} &= \Big\langle y^*, Tx\Big\rangle_{Y^* \times Y} = \Big\langle T^* y^*, x\Big\rangle_{X^* \times X} \\ &= \Big\langle \mathcal{I}_X x, T^* y^*\Big\rangle_{X^{**} \times X^*} = \Big\langle (T^*)^*(\mathcal{I}_X x), y^*\Big\rangle_{Y^{**} \times Y^*}.\end{aligned}$$

10.3. Dual operators and invertibility

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed $\mathbb{K}$-vector spaces (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) and $T \in L(X, Y)$. Prove the following.

(a) If T is an isomorphism with $T^{-1} \in L(Y, X)$, then T^* is an isomorphism.

Solution: The dual operator T^* of any $T \in L(X, Y)$ with $T^{-1} \in L(Y, X)$ is invertible according to Exercise 10.2(c) and its inverse is $(T^*)^{-1} = (T^{-1})^*$. Moreover, the assumption $T^{-1} \in L(Y, X)$ implies $(T^{-1})^* \in L(X^*, Y^*)$. Hence, T^* is an isomorphism.

(b) If T is an isometric isomorphism, then T^* is an isometric isomorphism.

Solution: If T is an isometric isomorphism, then T^* is an isomorphism by (a) and

$$\begin{aligned}\|T^*y^*\|_{X^*} &= \sup_{\|x\|_X \leq 1} \left|\langle T^*y^*, x\rangle_{X^* \times X}\right| = \sup_{\|x\|_X \leq 1} \left|\langle y^*, Tx\rangle_{Y^* \times Y}\right| \\ &= \sup_{\|y\|_Y \leq 1} |\langle y^*, y\rangle_{Y^* \times Y}| = \|y^*\|_{Y^*} \quad \text{for all } y^* \in Y^*.\end{aligned}$$

(c) If X and Y are both reflexive, then the reverse implications of (a) and (b) hold.

Solution: If X and Y are reflexive, $\mathcal{I}_X \colon X \to X^{**}$ and $\mathcal{I}_Y \colon Y \to Y^{**}$ are bijective isometries. If T^* is an (isometric) isomorphism, then Exercise 10.2 and (b) imply that $(T^*)^*$ is an (isometric) isomorphism. Applying Exercise 10.2(d), we see that the same holds for

$$T = \mathcal{I}_Y^{-1} \circ (T^*)^* \circ \mathcal{I}_X.$$

(d) If $(X, \|\cdot\|_X)$ is a reflexive Banach space isomorphic to the normed space $(Y, \|\cdot\|_Y)$, then Y is reflexive.

Solution: Since X is reflexive by assumption, $\mathcal{I}_X$ is an isomorphism. Suppose, $T \colon X \to Y$ is an isomorphism. Applying part (b) twice, $(T^*)^*$ is an isomorphism. Moreover,

$$\mathcal{I}_Y = (T^*)^* \circ \mathcal{I}_X \circ T^{-1}$$

according to Exercise 10.2(d). Since $\mathcal{I}_Y$ is a composition of isomorphisms, Y is reflexive.

10.4. Invariant measures again

Let (K, d) be a non-empty compact metric space and let $T \in L(C(K, \mathbb{R}), C(K, \mathbb{R}))$ satisfy

- $T\mathbf{1} = \mathbf{1}$, where $\mathbf{1} := (K \ni x \mapsto 1 \in \mathbb{R}) \in C(K, \mathbb{R})$ and
- $Tf \geq 0$ for all $f \in C(K, \mathbb{R})$ with $f \geq 0$.

(a) Prove for all $n \in \mathbb{N}$ that the mapping $S_n \colon \mathcal{P}(K) \to \mathcal{P}(K)$, defined via

$$\int_K f \, d(S_n \nu) = \frac{1}{n} \sum_{k=0}^{n-1} \int_K T^k f \, d\nu \quad \text{for all } f \in C(K, \mathbb{R}), \nu \in \mathcal{P}(K),$$

is indeed well-defined.

Solution: Let $n \in \mathbb{N}$, $\nu \in \mathcal{P}(K)$ be fixed. Note that

$$C(K, \mathbb{R}) \ni f \mapsto \frac{1}{n} \sum_{k=0}^{n-1} \int_K T^k f \, d\nu \in \mathbb{R}$$

is a positive linear functional which maps $\mathbf{1}$ to 1. The Riesz–Markov–Kakutani theorem thus implies that there exists a Borel probability measure μ such that

$$\int_K f \, d\mu = \frac{1}{n} \sum_{k=0}^{n-1} \int_K T^k f \, d\nu \quad \text{for all } f \in C(K, \mathbb{R}).$$

(Positivity and linearity imply that μ is a positive finite Borel regular measure, the fact that $\mathbf{1}$ is mapped to 1 implies that μ is a probability measure.)

(b) Show for all $\nu \in \mathcal{P}(K)$ that there exist $(n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ with $n_k \nearrow \infty$ as $k \to \infty$ and $\mu \in \mathcal{P}(K)$ such that

$$\int_K f \, d\mu = \lim_{k \to \infty} \int_K f \, d(S_{n_k} \nu) \quad \text{for all } f \in C(K, \mathbb{R}).$$

Solution: With $\mathcal{M}(K)$ denoting the signed Borel regular measures (equipped with the total variation norm), let $J \colon \mathcal{M}(K) \to (C(K, \mathbb{R}))^*$ be the isomorphism provided by the Riesz–Markov–Kakutani theorem, that is,

$$[J(\xi)](f) = \int_K f \, d\xi \quad \text{for all } f \in C(K, \mathbb{R}), \xi \in \mathcal{M}(K).$$

Let $\nu \in \mathcal{P}(K)$ be fixed. The measures $(S_n \nu)_{n \in \mathbb{N}} \subseteq \mathcal{P}(K)$ constructed in (a) satisfy that $\sup_{n \in \mathbb{N}} \| J(S_n \nu) \|_{C(K, \mathbb{R})^*} = \sup_{n \in \mathbb{N}} \| S_n \nu \|_{\mathcal{M}(K)} = \sup_{n \in \mathbb{N}} (S_n \nu)(K) = 1$. The Banach–Alaoglu theorem (and the fact that $C(K, \mathbb{R})$ is separable) ensure that there exist $(n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ with $n_k \nearrow \infty$ as $k \to \infty$ and a functional $\Phi \in C(K, \mathbb{R})^*$ such that $J(S_{n_k} \nu) \xrightarrow{\text{w}^*} \Phi$. The Riesz–Markov–Kakutani theorem thus implies that there exists $\mu \in \mathcal{M}(K)$ such that $\Phi = J(\mu)$, i.e.,

$$\lim_{k \to \infty} \int_K f \, d(S_{n_k} \nu) = \lim_{k \to \infty} [J(S_{n_k} \nu)](f) = \Phi(f) = [J(\mu)](f) = \int_K f \, d\mu.$$

Since Φ is positive and satisfies $\Phi \mathbf{1} = 1$, we obtain $\mu \in \mathcal{P}(K)$.

(c) Let $\nu, \mu \in \mathcal{P}(K)$ and $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ satisfy $n_k \nearrow \infty$ and $\int_K f\,d(S_{n_k}\nu) \to \int_K f\,d\mu$ as $k \to \infty$. Infer that

$$\int_K Tf\,d\mu = \int_K f\,d\mu \quad \text{for every } f \in C(K,\mathbb{R}).$$

Solution: Note first that it holds for all $k \in \mathbb{N}$, $f \in C(K,\mathbb{R})$ that

$$\begin{aligned}\left|\int_K Tf\,d(S_{n_k}\nu) - \int_K f\,d(S_{n_k}\nu)\right| &= \left|\frac{1}{n_k}\sum_{j=0}^{n_k-1}\int_K T^jTf\,d\nu - \frac{1}{n_k}\sum_{j=0}^{n-1}\int_K T^jf\,d\nu\right| \\ &= \frac{1}{n_k}\left|\int_K T^{n_k}f\,d\nu - \int_K f\,d\nu\right| \leq \frac{2}{n_k}\|f\|_{C(K,\mathbb{R})}.\end{aligned}$$

Passing to the limits as $k \to \infty$, we obtain that $\int_K Tf\,d\mu = \int_K f\,d\mu$ for every $f \in C(K,\mathbb{R})$.

(d) Prove for every $f \in C(K,\mathbb{R})$ with $Tf = f$ and $f \neq 0$ that there exists $\mu \in \mathcal{P}(K)$ satisfying

- $\int_K f\,d\mu \neq 0$ and
- $\int_K Tg\,d\mu = \int_K g\,d\mu$ for all $g \in C(K,\mathbb{R})$

Solution: Let $f \in C(K,\mathbb{R})$ with $Tf = f$ and $f \neq 0$. Then there exists $\nu \in \mathcal{P}(K)$ with $\int_K f\,d\nu \neq 0$ (e.g., $\nu = \delta_x$ for $x \in K$ with $f(x) \neq 0$). According to (b), there exist $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$, $\mu \in \mathcal{P}(K)$ satisfying $n_k \nearrow \infty$ and $J(S_{n_k}\nu) \xrightarrow{\text{w}^*} J(\mu)$ in $(C(K,\mathbb{R}))^*$ as $k \to \infty$. According to (c), we have that $T_\#\mu = \mu$. Finally, note that

$$\int_K f\,d(S_k\nu) = \int_K f\,d\nu \quad \text{for all } k \in \mathbb{N}$$

and therefore $\int_K f\,d\nu = \int_K f\,d\mu$.

(e) Solve Problem 9.5 (*Invariant measures à la Krylov–Bogolioubov*) again using (d).

Solution: With $\varphi \in C(K,K)$ (formerly called T in Problem 9.5), associate $T \in L(C(K,\mathbb{R}), C(K,\mathbb{R}))$ defined via

$$Tf = f \circ \varphi \quad \text{for every } f \in C(K,\mathbb{R}).$$

Note that T satisfies $T\mathbf{1} = \mathbf{1}$ and $Tf \geq 0$ for every $f \in C(K,\mathbb{R})$ with $f \geq 0$. Part (d) assures that there exists $\mu \in \mathcal{P}(K)$ satisfying for all $f \in C(K,\mathbb{R})$ that

$$\int_K f\,d\mu = \int_K Tf\,d\mu = \int_K f \circ \varphi\,d\mu.$$

(For the fixed point of T – denoted as f in (d) – we can take $\mathbf{1} \in C(K,\mathbb{R})$.)

10.5. Von Neumann's ergodic theorem

Let $(H, \langle\cdot,\cdot\rangle)$ be a $\mathbb{K}$-Hilbert space (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$), let T be a continuous linear operator on H with $\|T\|_{L(H,H)} \leq 1$, let $U := \ker(I - T)$ (with $I = (H \ni x \mapsto x \in H) \in L(H,H)$ being the identity operator), let P_U denote the orthogonal projection onto U and let $S_n := \frac{1}{n}\sum_{k=0}^{n-1} T^k$ for every $n \in \mathbb{N}$. Our goal is to show that

$$\limsup_{n\to\infty} \|S_n x - P_U x\|_H = 0 \quad \text{for all } x \in H.$$

For this, we recommend to proceed along the following steps:

(a) For all $x \in H$, we have $Tx = x$ if and only if $T^*x = x$.

Solution: "($\Rightarrow$)": Since $\|T^*\|_{L(H,H)} = \|T\|_{L(H,H)} \leq 1$, we have for all $x \in U$ (i.e., $x \in H$ with $Tx = x$) that

$$\|x\|_H \|T^*x\|_H \geq \langle x, T^*x\rangle = \langle Tx, x\rangle = \|x\|_H^2 \geq \|x\|_H \|T^*x\|_H, \tag{2}$$

which implies that $\|T^*x\|_H = \|x\|_H$ for all $x \in U$ (as well as $\langle Tx, x\rangle = \langle x, T^*x\rangle = \|x\|_H^2$ for all $x \in U$). Hence, we have for all $x \in U$ that

$$\|T^*x - x\|_H^2 = \|T^*x\|_H^2 - 2\operatorname{Re}\langle x, T^*x\rangle + \|x\|_H^2 = \|x\|_H^2 - 2\|x\|_H^2 + \|x\|_H^2 = 0.$$

Thus, $\ker(I - T) \subseteq \ker(I - T^*)$.

"($\Leftarrow$)": As $T^* \in L(H,H)$ also satisfies $\|T^*\|_{L(H,H)} \leq 1$, the argument above shows for all $x \in \ker(I - T^*)$ that $T^{**}x = x$. Since $T^{**} = T$ for every bounded linear operator on a Hilbert space, we have that $\ker(I - T) \supseteq \ker(I - T^*)$.

(b) $U^\perp = \overline{\operatorname{im}(I - T)}$.

Solution: We know from (a) that $U = \ker(I - T) = \ker(I - T^*)$. Hence, it holds that

$$U^\perp = (\ker(I - T^*))^\perp = (\operatorname{im}(I - T)^\perp)^\perp = \overline{\operatorname{im}(I - T)}.$$

(c) $\lim_{n\to\infty} S_n x = x$ for all $x \in U$ and $\lim_{n\to\infty} S_n x = 0$ for all $x \in U^\perp$.

Solution: For every $x \in U$, we have $Tx = x$, hence $S_n x = x$ for all $n \in \mathbb{N}$ and therefore $\limsup_{n\to\infty} \|S_n x - x\|_H = 0$. For every $x \in \operatorname{im}(I - T)$, there exists $y \in H$ such that $x = (I - T)y$. Hence, it holds for all $n \in \mathbb{N}$ that

$$\begin{aligned}
\limsup_{n\to\infty} \|S_n x\|_H &= \limsup_{n\to\infty} \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k (y - Ty) \right\|_H \\
&= \limsup_{n\to\infty} \left\| \frac{1}{n}(y - T^n y) \right\|_H \leq \limsup_{n\to\infty} \frac{2\|y\|_H}{n} = 0.
\end{aligned}$$

For every $x \in \overline{\operatorname{im}(I-T)}$, there is a sequence $(z_n)_{n\in\mathbb{N}} \subseteq \operatorname{im}(I-T)$ converging to x as $n \to \infty$ and since $S_n y_k \to 0$ as $n \to \infty$ for every $k \in \mathbb{N}$, we get that

$$\begin{aligned}\limsup_{n\to\infty} \|S_n x\|_H &\leq \limsup_{n\to\infty} \left[\|S_n x - S_n y_k\|_H + \|S_n y_k\|_H\right] \\ &= \limsup_{n\to\infty} \|S_n x - S_n y_k\|_H \leq \limsup_{n\to\infty} \left[\|S_n\|_{L(H,H)}\|x-y_k\|_H\right] \\ &\leq \|x-y_k\|_H \quad \text{for all } k \in \mathbb{N}.\end{aligned}$$

Hence, $\limsup_{n\to\infty} \|S_n x\|_H = 0$ for every $x \in \overline{\operatorname{im}(I-T)} = U^\perp$. To come full circle, note that every $x \in H$ can be written as $x = (x - P_U x) + P_U x$, where $x - P_U x \in U^\perp$ and $P_U x \in U$, and therefore, we obtain for every $x \in H$ that $S_n x \to P_U x$ as $n \to \infty$ because $S_n(x - P_U x) \to 0$ and $S_n P_U x \to P_U x$ as $n \to \infty$.

10.6. Von Neumann again

Let $(X, \|\cdot\|_X)$ be a reflexive space, let $T\colon X \to X$ be a continuous linear operator satisfying $\sup_{n\in\mathbb{N}_0}\|T^n\|_{L(X,X)} < \infty$, let $U := \ker(I-T)$ and let $S_n := \frac{1}{n}\sum_{k=0}^{n-1} T^k$ for every $n \in \mathbb{N}$.

(a) Show that $Y := \{x \in X \mid \lim_{n\to\infty} S_n x \text{ exists}\}$ is a closed subspace of X.

Solution: Clearly, $0 \in Y$ and for all $\alpha \in \mathbb{K}$, $x_1, x_2 \in Y$ it holds that $\alpha x_1 + x_2 \in Y$ since $\lim_{n\to\infty} S_n(\alpha x_1 + x_2) = \alpha \lim_{n\to\infty} S_n x_1 + \lim_{n\to\infty} S_n x_2$. The only issue left is the closedness of Y. For this, let $(x_n)_{n\in\mathbb{N}} \subseteq Y$ be a sequence converging to $x_\infty \in X$, i.e., $\limsup_{n\to\infty}\|x_n - x_\infty\|_X = 0$, and let $(y_n)_{n\in\mathbb{N}} \subseteq X$ denote the limits, i.e., $y_n = \lim_{k\to\infty} S_k x_n$ for every $n \in \mathbb{N}$. We are going to show that $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Note that for all $m, n \in \mathbb{N}$, it holds that

$$\begin{aligned}\|y_m - y_n\|_X &\leq \limsup_{k\to\infty}\Big[\|y_m - S_k x_m\|_X + \|S_k x_m - S_k x_n\|_X + \|S_k x_n - y_n\|_X\Big] \\ &\leq \limsup_{k\to\infty} \|y_m - S_k x_m\|_X + \sup_{l\in\mathbb{N}_0}\|T^l\|_{L(X,X)}\|x_m - x_n\|_X + \limsup_{k\to\infty}\|S_k x_n - y_n\|_X \\ &\leq \sup_{l\in\mathbb{N}_0}\|T^l\|_{L(X,X)}\|x_m - x_n\|_X.\end{aligned}$$

Since $(x_n)_{n\in\mathbb{N}} \subseteq X$ is a Cauchy sequence, it follows that $(y_n)_{n\in\mathbb{N}}$ is Cauchy, too. Denoting $y_\infty = \lim_{n\to\infty} y_n$, it just remains to show that $\lim_{n\to\infty} S_n x_\infty = y_\infty$. For this, note that for all $n, k \in \mathbb{N}$ it holds that

$$\begin{aligned}\|S_n x_\infty - y_\infty\|_X &\leq \|S_n x_\infty - S_n x_k\|_X + \|S_n x_k - y_k\|_X + \|y_k - y_\infty\|_X \\ &\leq \sup_{l\in\mathbb{N}_0}\|T^l\|_{L(X,X)}\|x_\infty - x_k\|_X + \|S_n x_k - y_k\|_X + \|y_k - y_\infty\|_X.\end{aligned}$$

Letting $n \to \infty$, we obtain for every $k \in \mathbb{N}$ that

$$\limsup_{n\to\infty} \|S_n x_\infty - y_\infty\|_X \leq \sup_{l\in\mathbb{N}_0} \|T^l\|_{L(X,X)} \|x_\infty - x_k\|_X + \|y_k - y_\infty\|_X.$$

As we let $k \to \infty$, we obtain $\limsup_{n\to\infty} \|S_n x_\infty - y_\infty\|_X = 0$. Hence, $x_\infty \in Y$ and Y is closed.

(b) Show that $P\colon Y \to X$, defined by $Px = \lim_{n\to\infty} S_n x$ is a continuous linear map satisfying $\operatorname{im}(P) = U \subseteq Y$, $\ker(P) = \overline{\operatorname{im}(I-T)}$, and $P^2 = P$. In particular, deduce that $Y = \ker(I-T) \oplus \overline{\operatorname{im}(I-T)}$.

Solution: P is clearly linear on Y. For all $x \in X$, $n \in \mathbb{N}$ it holds that

$$\left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k x \right\|_X \leq \sup_{k\in\mathbb{N}_0} \|T^k\|_{L(X,X)} \|x\|_X.$$

It follows that $\|Px\|_X \leq \sup_{k\in\mathbb{N}_0} \|T^k\|_{L(X,X)} \|x\|_X$ for all $x \in Y$. Hence, P is continuous. Moreover, note that for all $n \in \mathbb{N}$ it holds that $(I-T)S_n = \frac{1}{n}(I - T^n) = S_n(I-T)$. Hence, for all $x \in Y$ we have that

$$(I-T)Px = \lim_{n\to\infty} (I-T)S_n x = \lim_{n\to\infty} \frac{1}{n}(x - T^n x) = 0$$

by $\sup_{k\in\mathbb{N}_0} \|T^k\|_{L(X,X)} < \infty$. Thus, $\operatorname{im}(P) \subseteq U$. On the other hand, for every $x \in U$, it holds $S_n x = x$ for all $n \in \mathbb{N}$ and therefore $x \in Y$, $Px = x$. Hence, $P(U) = \operatorname{im}(P)$ and $P^2 x = Px$ for all $x \in Y$. Finally, for every $x \in \operatorname{im}(I-T)$, there exists $y \in X$ with $x = y - Ty$. Hence, for every $x \in \operatorname{im}(I-T)$, we have

$$S_n x = S_n(I-T)y = \frac{1}{n}(y - T^n y) \to 0 \quad \text{as } n \to \infty,$$

that is, $x \in Y$ and $Px = 0$. As Y is closed and P is continuous, $\overline{\operatorname{im}(I-T)} \subseteq \ker(P)$. On the other hand, if $x \in \ker(P)$, then

$$x - \frac{1}{n} \sum_{k=0}^{n-1} T^k x = \frac{1}{n} \sum_{k=0}^{n-1} (x - T^k x) \in \operatorname{im}(I-T) \quad \text{for all } n \in \mathbb{N},$$

and since the left hand side tends to x as $n \to \infty$, it follows that $x \in \overline{\operatorname{im}(I-T)}$. Since $U = \operatorname{im}(P)$ and $\overline{\operatorname{im}(I-T)} = \ker(P)$ and since $P^2 x = Px$ for all $x \in Y$, the mapping

$$Y \ni x \mapsto (Px, x - Px) \in \operatorname{im}(P) \times \ker(P)$$

is a (linear) isomorphism (of Banach spaces), cp. also the solution of Problem 6.1. In other words, $Y = \operatorname{im}(P) \oplus \ker(P) = \ker(I-T) \oplus \overline{\operatorname{im}(I-T)}$.

(c) Show for every $x^* \in Y^\perp$ that $T^*x^* = x^*$ and $x^* \in U^\perp$.

Solution: For every $x^* \in Y^\perp$ it holds – since $U \subseteq Y$ and $\operatorname{im}(I-T) \subseteq Y$ according to (b) – in particular that $x^* \in U^\perp$ and $x^* \in \operatorname{im}(I-T)^\perp$. The latter implies for all $x \in X$, $x^* \in Y^\perp$ that $x^*(x - Tx) = 0$, i.e., $(T^*x^*)(x) = x^*(Tx) = x^*(x)$, resulting in $T^*x^* = x^*$.

(d) Show for every $x \in X$ that $U \cap \overline{\operatorname{conv}}(\{T^k x\colon k \in \mathbb{N}_0\}) \neq \emptyset$.

Solution: For every $x \in X$ it holds that $(S_n x)_{n\in\mathbb{N}} \subseteq X$ is a bounded sequence. Since X is assumed to be reflexive, there exist $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ with $n_k \nearrow \infty$ as $k \to \infty$ and $y_\infty \in X$ such that $S_{n_k}x \xrightharpoonup{\text{w}} y_\infty$ as $k \to \infty$. The Banach–Mazur theorem (or, eventually, the Hahn–Banach theorem) ensures that $y_\infty \in \overline{\operatorname{conv}}(\{S_n x\colon n \in \mathbb{N}\})$ which – as $S_n x \in \operatorname{conv}(\{T^k x\colon k \in \mathbb{N}_0\})$ for every $n \in \mathbb{N}$ – implies that $y_\infty \in \overline{\operatorname{conv}}(\{T^k x\colon k \in \mathbb{N}_0\})$. Moreover, for all $x^* \in X^*$ it holds that

$$\begin{aligned}
|\langle x^*, (I-T)y_\infty\rangle_{X^*\times X}| &= |\langle x^* - T^*x^*, y_\infty\rangle_{X^*\times X}| \\
&= \lim_{k\to\infty} |\langle x^* - T^*x^*, S_{n_k}x\rangle_{X^*\times X}| \\
&= \lim_{k\to\infty} |\langle x^*, (I-T)S_{n_k}x\rangle_{X^*\times X}| \\
&= \lim_{k\to\infty} \left|\left\langle x^*, \frac{1}{n_k}(x - T^{n_k}x)\right\rangle_{X^*\times X}\right| \\
&\leq \limsup_{k\to\infty} \left[\frac{2}{n_k}\|x^*\|_{X^*}\|x\|_X \sup_{l\in\mathbb{N}_0}\|T^l\|_{L(X,X)}\right] = 0,
\end{aligned}$$

which implies that $(I-T)y_\infty = 0$, i.e., $y_\infty \in \ker(I-T) = U$.

(e) Show that $Y = X$.

Solution: Assume that there exists $x \in X \setminus Y = X \setminus \overline{Y}$. By the Hahn–Banach theorem, there exists $x^* \in Y^\perp$ with $x^*(x) = 1$. According to (c), $T^*x^* = x^*$ and $x^* \in U^\perp$. On the other hand, for all $n \in \mathbb{N}_0$, $\lambda_0, \lambda_1, \ldots, \lambda_n \in [0,1]$ with $\sum_{i=0}^n \lambda_i = 1$ it holds that

$$\left\langle x^*, \sum_{i=0}^n \lambda_i T^i x\right\rangle_{X^*\times X} = \sum_{i=0}^n \lambda_i \langle (T^*)^i x^*, x\rangle_{X^*\times X} = \sum_{i=0}^n \lambda_i \langle x^*, x\rangle_{X^*\times X} = \sum_{i=0}^n \lambda_i = 1.$$

Hence, x^* is constantly equal to 1 on the set $\operatorname{conv}(\{T^k x\colon k \in \mathbb{N}_0\})$, and – by continuity – also on the set $\overline{\operatorname{conv}}(\{T^k x\colon k \in \mathbb{N}_0\})$. Therefore, as $U \cap \overline{\operatorname{conv}}(\{T^k x\colon k \in \mathbb{N}_0\}) \neq \emptyset$ by (d), there exists $y \in U$ with $x^*(y) = 1$. This contradicts $x^* \in U^\perp$. Thus, $X = Y$. By definition of Y, we obtain for every $x \in X$ that $S_n x$ converges (strongly) as $n \to \infty$. Moreover, P, the mapping associating to $x \in X$ the limit of $S_n x$ as $n \to \infty$, is a projection.