### 11.1. Compact operators

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces. We denote by

$$K(X,Y) = \{T \in L(X,Y) \mid \overline{T(B_1(0))} \subseteq Y \text{ compact}\}$$

the set of compact operators between $X$ and $Y$. Prove the following statements.

**(a)** $T \in L(X,Y)$ is a compact operator if and only if every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $(Tx_{n_k})_{k\in\mathbb{N}}$ is convergent in $Y$.

**Solution:** "($\Rightarrow$)": Let $T \in L(X,Y)$ be a compact operator. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in $X$. Then there exists $M \in (0,\infty)$ such that $\|x_n\|_X < M$ for all $n \in \mathbb{N}$. In particular, $\frac{1}{M}x_n \in B_1(0) \subseteq X$ and $\frac{1}{M}Tx_n \in T(B_1(0))$ for every $n \in \mathbb{N}$. Since $\overline{T(B_1(0))} \subseteq Y$ is compact (and, in $(Y, \|\cdot\|_Y)$, compact $\Leftrightarrow$ sequentially compact), a subsequence $(\frac{1}{M}Tx_{n_k})_{k\in\mathbb{N}}$ converges in $Y$. Hence, $(Tx_{n_k})_{k\in\mathbb{N}}$ is also a convergent sequence.

"($\Leftarrow$)": Conversely, let $(y_n)_{n\in\mathbb{N}}$ be any sequence in $\overline{T(B_1(0))}$. For every $n \in \mathbb{N}$ there exists $z_n \in T(B_1(0))$ such that $\|y_n - z_n\|_Y \leq \frac{1}{n}$. Since there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $B_1(0) \subseteq X$ such that $Tx_n = z_n$, a subsequence $(z_{n_k})_{k\in\mathbb{N}}$ converges to some $z_\infty \in Y$ as $k \to \infty$ by assumption. Since

$$\begin{aligned}\limsup_{k\to\infty} \|y_{n_k} - z_\infty\|_Y &\leq \limsup_{k\to\infty} \left[\|y_{n_k} - z_{n_k}\| + \|z_{n_k} - z_\infty\|_Y\right] \\ &\leq \limsup_{k\to\infty} \left[\frac{1}{n_k} + \|z_{n_k} - z_\infty\|_Y\right] = 0,\end{aligned}$$

we conclude that a subsequence of $(y_n)_{n\in\mathbb{N}}$ converges. Being closed, $\overline{T(B_1(0))}$ must contain the limit $z_\infty$ which proves that $\overline{T(B_1(0))}$ is sequentially compact. By equivalence of compactness and sequential compactness in metric spaces, we obtain that $\overline{T(B_1(0))}$ is compact. Hence, $T$ is a compact operator.

**(b)** If $(Y, \|\cdot\|_Y)$ is complete, then $K(X,Y)$ is a closed subspace of $L(X,Y)$.

**Solution:** Part (a) and linearity of the limit imply that the set of compact operators $K(X,Y) \subseteq L(X,Y)$ is a linear subspace. To prove that this subspace is closed, let $(T_k)_{k\in\mathbb{N}}$ be a sequence in $K(X,Y)$ such that $\|T_k - T\|_{L(X,Y)} \to 0$ for some $T \in L(X,Y)$ as $k \to \infty$. To show $T \in K(X,Y)$, consider a bounded sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ and choose the nested, unbounded subsets $\mathbb{N} \supseteq \Lambda_1 \supseteq \Lambda_2 \supseteq \ldots$ such that $(T_k x_n)_{n\in\Lambda_k}$ is convergent in $Y$ with limit point $y_k \in Y$. This is possible by (i) since $T_k$ is a compact operator for every $k \in \mathbb{N}$. Let $\Lambda \subseteq \mathbb{N}$ be the corresponding diagonal sequence (i.e., the $k^{th}$ number in $\Lambda$ is the $k^{th}$ number in $\Lambda_k$). By continuity of $\|\cdot\|_Y$, we can estimate

$$\|y_k - y_m\|_Y = \lim_{\Lambda \ni n\to\infty} \|T_k x_n - T_m x_n\|_Y \leq \|T_k - T_m\|_{L(X,Y)} \sup_{n\in\Lambda} \|x_n\|_X$$

for any $k, m \in \mathbb{N}$. Since $(T_k)_{k\in\mathbb{N}}$ is convergent in $L(X,Y)$, we conclude that $(y_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $Y$. Since $(Y, \|\cdot\|_Y)$ is assumed to be complete, $y_k \to y_\infty$ for some $y_\infty \in Y$ as $k \to \infty$. It then suffices to prove the following.

*Claim.* The sequence $(Tx_n)_{n\in\Lambda}$ converges to $y_\infty$.

*Proof.* Let $\varepsilon \in (0,\infty)$. Choose a fixed $\kappa \in \mathbb{N}$ such that $\|T - T_\kappa\|_{L(X,Y)} < \varepsilon$ and $\|y_\kappa - y_\infty\|_Y \le \varepsilon$. Since $T_\kappa x_n \to y_\kappa$ as $\Lambda \ni n \to \infty$, there exists $N \in \Lambda$ such that for every $\Lambda \ni n \ge N$ it holds that $\|T_\kappa x_n - y_\kappa\| \le \varepsilon$. Finally, the claim follows from the estimate

$$\begin{aligned}\|Tx_n - y_\infty\|_Y &\le \|Tx_n - T_\kappa x_n\|_Y + \|T_\kappa x_n - y_\kappa\|_Y + \|y_\kappa - y_\infty\|_Y \\ &\le \|T - T_\kappa\|_{L(X,Y)} \sup_{m\in\Lambda} \|x_m\|_X + \|T_\kappa x_n - y_\kappa\|_Y + \|y_\kappa - y_\infty\|_Y \\ &< 2\varepsilon + \varepsilon \sup_{m\in\Lambda} \|x_m\|_X,\end{aligned}$$

which holds for every $\Lambda \ni n \ge N$. Since $\varepsilon \in (0,\infty)$ was arbitrary, the claim follows. $\square$

**(c)** Let $T \in L(X,Y)$. If its range $T(X) \subseteq Y$ is finite-dimensional, then $T \in K(X,Y)$.

**Solution:** The image of $B_1(0)$ under $T \in L(X,Y)$ is bounded. If $T(X) \subseteq Y$ is of finite dimension, then so is $\overline{T(X)} = T(X)$, and $\overline{T(B_1(0))}$ is compact as a bounded, closed subset of $\overline{T(X)}$.

**(d)** Let $T \in L(X,Y)$ and $S \in L(Y,Z)$. If $T$ or $S$ is a compact operator, then $S \circ T$ is a compact operator.

**Solution:** Suppose $T$ is a compact operator. Then, a subsequence $(Tx_{n_k})_{k\in\mathbb{N}}$ is convergent in $Y$ by (a). Since $S$ is continuous, $(STx_{n_k})_{k\in\mathbb{N}}$ is convergent in $Z$, which by (a) proves that $S \circ T$ is a compact operator.

Suppose $S$ is a compact operator. Since $T$ is continuous, the sequence $(Tx_n)_{n\in\mathbb{N}}$ is bounded in $Y$. Then, a subsequence $(STx_{n_k})_{k\in\mathbb{N}}$ is convergent in $Z$ by (a), which again proves that $S \circ T$ is a compact operator.

**(e)** If $X$ is reflexive, then any operator $T \in L(X,Y)$ which maps weakly convergent sequences to strongly convergent sequences, that is

$$x_n \xrightarrow{\mathrm{w}} x \text{ in } X \quad \Longrightarrow \quad Tx_n \to x \text{ in } Y,$$

is a compact operator.

**Solution:** Let $(x_n)_{n\in\mathbb{N}}$ be any bounded sequence in $X$. Since $X$ is reflexive, a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converges weakly in $X$ by the Eberlein–Šmulian theorem. Then,

$(Tx_{n_k})_{k\in\mathbb{N}}$ is norm-convergent in $Y$ by assumption and (a) implies that $T$ is a compact operator.

### 11.2. Schauder's theorem

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and let $T \in L(X,Y)$ be a bounded linear operator. Prove that $T$ is compact if and only if $T^*$ is compact.

**Solution:** "($\Rightarrow$)": Let $T \in L(X,Y)$ be compact. This implies in particular that $K := \overline{T(B)} \subseteq Y$ is compact, where $B \subseteq X$ denotes the closed unit ball. Moreover, let $B^* \subseteq Y^*$ denote the closed unit ball in $Y^*$ and let $(x_n^*)_{n\in\mathbb{N}} \subseteq T^*(B^*)$ be an arbitrary sequence. Our goal is to show that $(x_n^*)_{n\in\mathbb{N}}$ possesses a converging subsequence. Let $(y_n^*)_{n\in\mathbb{N}} \subseteq B^*$ be a sequence s.t. $x_n^* = T_n^* y_n^*$ for all $n \in \mathbb{N}$ and let $\mathcal{F} := \{y_n^*|_K : n \in \mathbb{N}\} \subseteq C(K,\mathbb{R})$ be the set of those continuous functions on $K$ obtained by restricting the elements of the sequence $(y_n^*)_{n\in\mathbb{N}}$ to $K$. On the one hand, $\mathcal{F}$ is bounded since (by density of $T(B)$ in $K$)

$$\sup_{f\in\mathcal{F}} \|f\|_{C(K,\mathbb{R})} = \sup_{f\in\mathcal{F}} \|f\|_{C(T(B),\mathbb{R})} \leq \sup_{n\in\mathbb{N}, x\in B} [\|y_n^*\|_{Y^*} \|Tx\|_Y] \leq \|T\|_{L(X,Y)} < \infty.$$

On the other hand, $\mathcal{F}$ is equi-continuous since, for all $y_1, y_2 \in K$, it holds that

$$\begin{aligned}
\sup_{f\in\mathcal{F}} |f(y_1) - f(y_2)| &\leq \sup_{y^*\in Y^*, \|y^*\|_{Y^*}\leq 1} |y^*(y_1) - y^*(y_2)| \\
&\leq \sup_{y^*\in Y^*, \|y^*\|_{Y^*}\leq 1} \|y^*\|_{Y^*} \|y_1 - y_2\|_Y = \|y_1 - y_2\|_Y.
\end{aligned}$$

The Arzéla–Ascoli theorem thus ensures that there exists a sequence $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ with $n_k \nearrow \infty$ as $k \to \infty$ such that $(y_{n_k}^*|_K)_{k\in\mathbb{N}} \subseteq C(K,\mathbb{R})$ is uniformly converging. This implies that

$$\begin{aligned}
&\limsup_{N\to\infty} \left[ \sup_{k,l\geq N} \|T^* y_{n_k}^* - T^* y_{n_l}^*\|_{X^*} \right] \\
&\leq \limsup_{N\to\infty} \left[ \sup_{k,l\geq N, x\in X, \|x\|_X\leq 1} |T^* y_{n_k}^*(x) - T^* y_{n_l}^*(x)| \right] \\
&= \limsup_{N\to\infty} \left[ \sup_{k,l\geq N, x\in X, \|x\|_X\leq 1} |y_{n_k}^*(Tx) - y_{n_l}^*(Tx)| \right] \\
&\leq \limsup_{N\to\infty} \left[ \sup_{k,l\geq N} \|y_{n_k}^*|_K - y_{n_l}^*|_K\|_{C(K,\mathbb{R})} \right] = 0,
\end{aligned}$$

i.e., that $(T^* y_{n_k}^*)_{k\in\mathbb{N}} \subseteq X^*$ is a Cauchy sequence and thus has a limit in $X^*$.

"($\Leftarrow$)": Let $T \in L(X,Y)$ and assume that $T^*$ is compact. Since $T^* \in L(Y^*, X^*)$, the previous argument implies that $T^{**} \in L(X^{**}, Y^{**})$ is compact. Let $(x_n)_{n\in\mathbb{N}} \subseteq X$ be an arbitrary sequence satisfying $\sup_{n\in\mathbb{N}} \|x_n\|_X \leq 1$. With $\iota_X\colon X \to X^{**}$ denoting the canonical embedding, we obtain that $(\iota_X(x_n))_{n\in\mathbb{N}} \subseteq X^{**}$ is a sequence with norms bounded by 1. Since $T^{**}$ is compact, there exists a sequence $(n_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ with $n_k \nearrow \infty$ as $k \to \infty$ such that $(T^{**}\iota_X(x_{n_k}))_{k\in\mathbb{N}} \subseteq Y^{**}$ is a converging sequence. The fact that $T^{**}\iota_X = \iota_Y T$ and the fact that $\iota_Y$ is an isometry imply that $(Tx_{n_k})_{k\in\mathbb{N}}$ is a converging sequence in $Y$.

### 11.3. Various notions of continuity – continued

Suppose $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are $\mathbb{K}$-Banach spaces (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$).

**(a)** Let $B^*$ be the closed unit ball in $Y^*$, equipped with the weak* topology. Prove that a bounded linear operator $A\colon X \to Y$ is compact if and only if $A^*|_{B^*}\colon B^* \to X^*$ is $\sigma(B^*, Y)$-$\|\cdot\|_{X^*}$-continuous (i.e., weak*-norm continuous).

**Solution:** If $A$ is compact then so is $A^*$ by Problem 11.2 (*Schauder's theorem*). Since $A^*$ is $\sigma(Y^*, Y)$-$\sigma(X^*, X)$-continuous and $B^*$ is weak*-compact (more precisely, compact w.r.t. the topology $\sigma(Y^*, Y)$), the set $A^*(B^*)$ is weak*-compact in $X^*$. As such, it is norm-closed and by compactness of $A^*$, it is also norm-compact. Let $O \subseteq X^*$ be an arbitrary but fixed norm-open set. We need to show that $(A^*|_{B^*})^{-1}(O) \in \sigma(B^*, Y)$. For this, let $y^* \in B^*$ with $A^*y^* \in O$ be arbitrary but fixed. There exists $\chi\colon A^*(B^*) \setminus O \to X$ such that for every $x^* \in A^*(B^*) \setminus O$ it holds that $|x^*(\chi(x^*)) - (A^*y^*)(\chi(x^*))| > 1$. Thus, $A^*(B^*) \setminus O$ can be covered by a union of elements of $\sigma(X^*, X)$:

$$A^*(B^*) \setminus O \subseteq \bigcup_{x^*\in A^*(B^*)\setminus O} \left\{\xi^* \in X^*\colon |\xi^*(\chi(x^*)) - x^*(\chi(x^*))| < \frac{1}{2}\right\}.$$

Since $A^*(B^*) \setminus O$ is – as norm-closed subset of a norm-compact set – norm-compact, there exist $n \in \mathbb{N}$, $x_1^*, \ldots, x_n^* \in X^*$ (and $x_1 = \chi(x_1^*), x_2 = \chi(x_2^*), \ldots, x_n = \chi(x_n^*) \in X$) such that

$$A^*(B^*) \setminus O \subseteq \bigcup_{i=1}^{n} \left\{\xi^* \in X^*\colon |x_i^*(x_i) - \xi^*(x_i)| < \frac{1}{2}\right\}.$$

Since $|(A^*y^*)(x_i) - x_i^*(x_i)| > 1$ for every $i \in \{1, 2, \ldots, n\}$, the above assures that

$$\begin{aligned} A^*(B^*) \cap O &\supseteq \bigcap_{i=1}^{n} \left\{\xi^* \in A^*(B^*)\colon |x_i^*(x_i) - \xi^*(x_i)| \geq \frac{1}{2}\right\} \\ &\supseteq \bigcap_{i=1}^{n} \left\{\xi^* \in A^*(B^*)\colon |\xi^*(x_i) - (A^*y^*)(x_i)| < \frac{1}{2}\right\}. \end{aligned}$$

This leads to

$$\begin{aligned}&\bigcap_{i=1}^{n}\left\{\upsilon^*\in B^*\colon |\upsilon^*(Ax_i)-y^*(Ax_i)|<\frac{1}{2}\right\}\\&=\bigcap_{i=1}^{n}\left\{\upsilon^*\in B^*\colon |(A^*\upsilon^*)(x_i)-(A^*y^*)(x_i)|<\frac{1}{2}\right\}\subseteq (A^*|_{B^*})^{-1}(O).\end{aligned}$$

Thus, $(A^*|_{B^*})^{-1}(O)\in\sigma(B^*,Y)$.

Conversely, if $A^*|_{B^*}$ is weak*-norm continuous, then – since $B^*$ is weak*-compact – $A^*(B^*)$ is norm-compact. Thus, $A^*$ is compact and by Problem 11.2 (*Schauder's theorem*), so is $A$.

**(b)** Suppose $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ are reflexive. A linear operator $A\colon X\to Y$ is compact if and only if $A|_B\colon B\to Y$ is $\sigma(B,X^*)$-$\|\cdot\|_Y$-continuous (i.e., weak–norm continuous).

**Solution:** Schauder's theorem implies that $A$ is compact if and only if $A^*\in L(Y^*,X^*)$ is compact. Part (a) implies on the other hand that this is equivalent to $A^{**}|_{B^{**}}\colon B^{**}\to Y^{**}$ being $\sigma(B^{**},X^*)$-$\|\cdot\|_{Y^{**}}$-continuous. The reflexivity of $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ implies that $A|_B\colon B\to Y$ is $\sigma(B,X^*)$-$\|\cdot\|_Y$-continuous if and only if $A^{**}|_{B^{**}}\colon B^{**}\to Y^{**}$ is $\sigma(B^{**},X^*)$-$\|\cdot\|_{Y^{**}}$-continuous.

### 11.4. Ehrling's lemma

Let $(X,\|\cdot\|_X)$, $(Y,\|\cdot\|_Y)$ and $(Z,\|\cdot\|_Z)$ be Banach spaces, let $T\in L(X,Y)$ be compact and let $J\in L(Y,Z)$ be injective. Prove that for every $\varepsilon\in(0,\infty)$, there exists $C\in[0,\infty)$ such that

$$\|Tx\|_Y\leq\varepsilon\|x\|_X+C\|JTx\|_Z\quad\text{for all } x\in X.$$

**Solution:** Assume for a contradiction that the claim is not true. Then there exist $\varepsilon\in(0,\infty)$ and $(x_n)_{n\in\mathbb{N}}\subseteq X$ such that

$$\|Tx_n\|_Y>\varepsilon\|x_n\|_X+n\|JTx_n\|_Z\quad\text{for all } n\in\mathbb{N}.$$

In particular, it holds for every $n\in\mathbb{N}$ that $Tx_n\neq 0$ so that the sequence $(x_n')_{n\in\mathbb{N}}\subseteq X$, given by $x_n'=\frac{x_n}{\|Tx_n\|_Y}$ for all $n\in\mathbb{N}$, is well-defined and satisfies

$$1=\|Tx_n'\|_Y>\varepsilon\|x_n'\|_X+n\|JTx_n'\|_Z\quad\text{for all } n\in\mathbb{N}.$$

This implies that, on the one hand, $(x_n')_{n\in\mathbb{N}}\subseteq X$ is bounded and, on the other hand, that $JTx_n'\to 0$ in $Z$ as $n\to\infty$. The boundedness of $(x_n')_{n\in\mathbb{N}}\subseteq X$ and the assumption that $T$ is compact imply – based on part (a) of Problem 11.1 (*Compact operators*) – that there exists a subsequence $(x'_{n_k})_{k\in\mathbb{N}}$ such that $(Tx'_{n_k})_{k\in\mathbb{N}} \subseteq Y$ converges to some limit $y \in Y$ as $k \to \infty$. The fact that $\|Tx'_n\|_Y = 1$ for every $n \in \mathbb{N}$ implies that $\|y\|_Y = 1$. The assumed continuity of $J$, on the other hand, implies that $JTx'_{n_k} \to Jy$ in $Z$ as $k \to \infty$. Since $JTx'_n \to 0$ as $n \to \infty$, we conclude that $Jy = 0$. By injectivity of $J$, we obtain $y = 0$. This, however, contradicts $\|y\|_Y = 1$, which we had already deduced above.

### 11.5. Integral operators

Let $m \in \mathbb{N}$ and let $\emptyset \neq \Omega \subseteq \mathbb{R}^m$ be a bounded open set. Given $k \in L^2(\Omega \times \Omega, \mathbb{C})$, consider the linear operator $K\colon L^2(\Omega, \mathbb{C}) \to L^2(\Omega, \mathbb{C})$ defined by

$$(Kf)(x) = \int_\Omega k(x,y) f(y)\, dy.$$

**(a)** Prove that $K$ is well-defined, i.e., $Kf \in L^2(\Omega, \mathbb{C})$ for any $f \in L^2(\Omega, \mathbb{C})$.

**Solution:** Let $f \in L^2(\Omega, \mathbb{C})$. Then Hölder's inequality and Tonelli's theorem imply

$$\begin{aligned}\int_\Omega |(Kf)(x)|^2\, dx &= \int_\Omega \left| \int_\Omega k(x,y) f(y)\, dy \right|^2 dx \leq \int_\Omega \left( \int_\Omega |k(x,y) f(y)|\, dy \right)^2 dx \\ &\leq \int_\Omega \left( \int_\Omega |k(x,y)|^2\, dy \right) \|f\|^2_{L^2(\Omega)}\, dx = \|k\|^2_{L^2(\Omega\times\Omega)} \|f\|^2_{L^2(\Omega)}.\end{aligned}$$

Since $k \in L^2(\Omega \times \Omega, \mathbb{C})$ by assumption, $\|Kf\|_{L^2(\Omega,\mathbb{C})} \leq \|k\|_{L^2(\Omega\times\Omega,\mathbb{C})} \|f\|_{L^2(\Omega,\mathbb{C})} < \infty$ follows.

**(b)** Prove that $K$ is a compact operator.

**Solution:** Being a Hilbert space, $L^2(\Omega, \mathbb{C})$ is reflexive. Part (e) of Problem 11.1 (*Compact operators*) implies that $K\colon L^2(\Omega, \mathbb{C}) \to L^2(\Omega, \mathbb{C})$ is a compact operator if $K$ maps weakly convergent sequences to norm-convergent sequences.

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^2(\Omega, \mathbb{C})$ such that $f_n \xrightharpoonup{\text{w}} f$ as $n \to \infty$ for some $f \in L^2(\Omega, \mathbb{C})$. Since $k \in L^2(\Omega \times \Omega, \mathbb{C})$, Fubini's theorem implies that $k(x, \cdot) \in L^2(\Omega, \mathbb{C})$ for almost every $x \in \Omega$. Weak convergence therefore implies

$$(Kf_n)(x) = \left\langle k(x,\cdot), f_n \right\rangle_{L^2(\Omega,\mathbb{C})} \xrightarrow{n\to\infty} \left\langle k(x,\cdot), f \right\rangle_{L^2(\Omega,\mathbb{C})} = (Kf)(x)$$

for almost every $x \in \Omega$. As weakly convergent sequence, $(f_n)_{n\in\mathbb{N}}$ is bounded: there exists $C \in \mathbb{R}$ such that $\|f_n\|_{L^2(\Omega,\mathbb{C})} \leq C$ for every $n \in \mathbb{N}$. By Hölder's inequality,

$$|(Kf_n)(x)| \leq \int_\Omega |k(x,y) f_n(y)|\, dy \leq \|k(x,\cdot)\|_{L^2(\Omega)} \|f_n\|_{L^2(\Omega)} \leq C \|k(x,\cdot)\|_{L^2(\Omega)}.$$

The assumption $k \in L^2(\Omega \times \Omega, \mathbb{C})$ and Fubini's theorem imply that (the equivalence class of) the function $x \mapsto C\|k(x,\cdot)\|_{L^2(\Omega,\mathbb{C})}$ is in $L^2(\Omega, \mathbb{C})$. Thus, $(Kf_n)(x)$ is dominated by a function in $L^2(\Omega, \mathbb{C})$. Since $(Kf_n)(x)$ converges pointwise for almost every $x \in \Omega$ and since $(Kf_n)$ is dominated by a function in $L^2(\Omega, \mathbb{C})$, Lebesgue's dominated convergence theorem implies $L^2$-convergence $\|Kf_n - Kf\|_{L^2(\Omega,\mathbb{C})} \to 0$ as $n \to \infty$.

**(c)** If, in addition, the kernel $k$ satisfies $k(x, y) = \overline{k(y, x)}$ for almost every $(x, y) \in \Omega \times \Omega$, prove that the operator $A\colon L^2(\Omega, \mathbb{C}) \to L^2(\Omega, \mathbb{C})$, defined by

$$Af = f - Kf,$$

is surjective if and only if it is injective.

**Solution:** For $f, g \in L^2(\Omega, \mathbb{C})$ using repeatedly Fubini's theorem we compute:

$$\begin{aligned}
(Kf, g)_{L^2} &= \int_\Omega Kf(x)\overline{g(x)}\,dx \\
&= \int_\Omega \left(\int_\Omega k(x, y)f(y)\,dy\right)\overline{g(x)}\,dx \\
&= \int_{\Omega\times\Omega} k(x, y)\overline{g(x)}f(y)\,dx\,dy \\
&= \int_\Omega f(y)\left(\int_\Omega k(x, y)\overline{g(x)}\,dx\right)dy \\
&= \int_\Omega f(y)\overline{\left(\int_\Omega \overline{k(x, y)}g(x)\,dx\right)}\,dy = (f, K^*g)_{L^2},
\end{aligned}$$

that is,

$$(K^*g)(x) = \int_\Omega \overline{k(y, x)}g(y)dy.$$

Hence, under the additional assumption that $k(x, y) = \overline{k(y, x)}$ for a.a. $x, y \in \Omega$, the bounded operator $K$ is self-adjoint. Therefore, the operator $A = (1 - K)\colon L^2(\Omega) \to L^2(\Omega)$ is also self-adjoint.

According to (b), $K$ is a compact operator, which implies that the operator $A = (1-K)$ has closed image $\operatorname{im}(A) \subseteq H$. According to Banach's closed range theorem, this is equivalent to $\operatorname{im}(A) = \ker(A^*)^\perp$. Since $A^* = A$, we conclude in our setting that

$$A \text{ surjective} \;\Leftrightarrow\; H = \operatorname{im}(A) = \ker(A)^\perp \;\Leftrightarrow\; \ker(A) = \{0\} \;\Leftrightarrow\; A \text{ injective}.$$

### 11.6. Integral operators again

Let $m \in \mathbb{N}$ and let $\emptyset \neq K \subseteq \mathbb{R}^m$ be a non-empty compact set. Given $k \in C(K \times K, \mathbb{R})$, consider the linear operator $T\colon C(K,\mathbb{R}) \to C(K,\mathbb{R})$ defined by

$$(Tf)(x) = \int_K k(x,y)f(y)\,dy \quad \text{for all } f \in C(K,\mathbb{R}).$$

**(a)** Prove that $T$ is well-defined, i.e., $Tf \in C(K,\mathbb{R})$ for every $f \in C(K,\mathbb{R})$.

**Solution:** Clearly, for every $x \in K$, $f \in C(K,\mathbb{R})$,

$$\int_K |k(x,y)||f(y)|\,dy \leq \|k\|_{C(K\times K,\mathbb{R})}\|f\|_{C(K,\mathbb{R})}|K| < \infty$$

Moreover, for all $(x_n)_{n\in\mathbb{N}} \subseteq K$, $x_\infty \in K$ with $x_n \to x_\infty$ in $K$ as $n \to \infty$, it holds by Lebesgue's dominated convergence theorem that

$$\limsup_{n\to\infty} \int_K |(k(x_n,y) - k(x_\infty,y))f(y)|\,dy = 0.$$

Hence, $Tf \in C(K,\mathbb{R})$ for every $f \in C(K,\mathbb{R})$.

**(b)** Prove that $T$ is a compact operator.

**Solution:** Let $\mathcal{F} = \{Tf \mid f \in C(K,\mathbb{R}), \|f\|_{C(K,\mathbb{R})} \leq 1\} \subseteq C(K,\mathbb{R})$. Then, clearly,

$$\sup_{g\in\mathcal{F}} \|g\|_{C(K,\mathbb{R})} \leq \|T\|_{L(C(K,\mathbb{R}),C(K,\mathbb{R}))} \leq |K|\|k\|_{C(K\times K,\mathbb{R})} < \infty,$$

that is, $\mathcal{F} \subseteq C(K,\mathbb{R})$ is bounded. For the application of the Arzéla–Ascoli theorem to be justified, it remains to show that $\mathcal{F}$ is equi-continuous. For this, note that $k$ is uniformly continuous on the compact set $K \times K$ and therefore, there exists $\delta\colon (0,\infty) \to (0,\infty)$ such that for all $\varepsilon \in (0,\infty)$, $(x_1,y_1),(x_2,y_2) \in K \times K$ with $\|x_1 - x_2\|_{\mathbb{R}^m} + \|y_1 - y_2\|_{\mathbb{R}^m} < \delta_\varepsilon$ it holds that $\|k(x_1,y_1) - k(x_2,y_2)\|_{\mathbb{R}^m} < \varepsilon$. Hence, for all $n \in \mathbb{N}$, $\varepsilon \in (0,\infty)$, $x_1, x_2 \in K$ with $\|x_1 - x_2\|_{\mathbb{R}^m} < \delta_\varepsilon$, we have that

$$\begin{aligned}
\sup_{g\in\mathcal{F}} |g(x_1) - g(x_2)| &= \sup_{f\in C(K,\mathbb{R}),\|f\|_{C(K,\mathbb{R})}\leq 1} |(Tf)(x_1) - (Tf)(x_2)| \\
&\leq \sup_{f\in C(K,\mathbb{R}),\|f\|_{C(K,\mathbb{R})}\leq 1} \int_K |k(x_1,y) - k(x_2,y)||f(y)|\,dy \\
&\leq \varepsilon|K|.
\end{aligned}$$

This establishes the equi-continuity of $\mathcal{F}$. Combining this with the previously proved boundedness of $\mathcal{F}$ and the Arzéla–Ascoli theorem, we obtain that $\overline{\mathcal{F}} \subseteq C(K,\mathbb{R})$ is compact, i.e., $T$ is a compact operator.

**(c)** If $k(x,y) = k(y,x)$ for all $x, y \in K$, prove that the operator $A \colon C(K,\mathbb{R}) \to C(K,\mathbb{R})$, defined by $Af = f - Tf$ for every $f \in C(K,\mathbb{R})$ is surjective if and only if it is injective.

**Solution:** By the fact that $\operatorname{im}(I-T)$ is closed as $T$ is compact, the closed range theorem ensures that $A = I - T$ is surjective if and only if $\ker(A^*) = \{0\}$, that is, if and only if $A^*$ is injective. By the Riesz–Markov–Kakutani representation theorem, we can (and will) identify $(C(K,\mathbb{R}))^*$ with the space $\mathcal{M}(K)$ of finite signed regular Borel measures on $K$. Also we will sloppily consider $A^*$ and $T^*$ as maps from $\mathcal{M}(K)$ to $\mathcal{M}(K)$. This being said, note that for all $\mu \in \mathcal{M}(K)$, $f \in C(K,\mathbb{R})$ it holds that

$$\begin{aligned}\langle T^*\mu, f\rangle = \langle \mu, Tf\rangle &= \int_K (Tf)(x)\,d\mu(x) = \int_K \int_K k(x,y) f(y)\,dy\,d\mu(x)\\ &= \int_K f(y) \int_K k(x,y)\,d\mu(x)\,dy,\end{aligned}$$

where the last term can be interpreted as the integration of $f$ against the signed measure with density $K \ni x \mapsto \int_K k(y,x)\,d\mu(y) \in \mathbb{R}$ w.r.t. the Lebesgue measure on $K$. In other words, for every $\mu \in \mathcal{M}(K)$ and every Borel-measurable set $A \subseteq K$, it holds that

$$(T^*\nu)(A) = \int_A \int_K k(y,x)\,d\mu(y)\,dx$$

For every $\mu \in \ker(A^*)$, it therefore has to hold that $\mu$ has a density $h$ w.r.t. the Lebesgue measure on $K$ and for a.e. $x \in K$, it holds that

$$h(x) = \int_K k(y,x)\,d\mu(y) = \int_K k(y,x) h(y)\,dy.$$

Now, by similar arguments as in part (a), the term on the right hand side is continuous w.r.t. $x$. Hence, we may assume that $h$ is continuous and satisfies $h(x) = \int_K k(y,x)h(y)\,dy$ for every $x \in K$. With the additional assumption that $k(y,x) = k(x,y)$ for all $x, y \in K$, we obtain that surjectivity of $A$ is equivalent to the equation $h = Kh$ having only the trivial solution, that is, to the injectivity of $A$.

*Remark.* In both 11.5.(c) and 11.6.(c), the symmetry assumption on $k$ is not really necessary. Riesz–Schauder theory ensures that $\dim(\ker(I-T)) = \dim(\ker(I-T^*)) = \operatorname{codim}(\operatorname{im}(I-T)) = \operatorname{codim}(\operatorname{im}(I-T^*))$.

### 11.7. A dual statement

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and let $T \colon D(T) \subseteq X \to Y$ be a densely defined closed linear operator. Prove that the following properties are equivalent:

(i) $T^*$ is surjective.

(ii) There exists $C \in [0, \infty)$ such that $\|u\|_X \leq C\|Tu\|_Y$ for all $u \in D(T)$.

(iii) $T$ is injective and has closed range.

**Solution:** "$(i) \Rightarrow (ii)$": Since $T$ is a closed linear operator and $\operatorname{im}(T^*) = X^*$ is closed, Banach's closed range theorem ensures that $\ker(T)^\perp = X^*$, i.e., $\ker(T) = \{0\}$. Hence, $T^{-1} \colon Y \to X$ is a well-defined closed operator. By the closed range theorem, $T^{-1}$ is continuous. Hence, there exists $C \in [0, \infty)$ such that for every $u \in D(T)$ it holds that $\|u\|_X = \|T^{-1}Tu\|_X \leq C\|Tu\|_Y$.

"$(ii) \Rightarrow (iii)$": From $\|u\|_X \leq C\|Tu\|_Y$ for all $u \in D(T)$ it follows for every $u \in \ker(T)$ immediately that $u = 0$ so that $T$ is injective. Moreover, let $(y_n)_{n\in\mathbb{N}} \subseteq \operatorname{im}(T)$ and $y_\infty \in Y$ satisfy $\limsup_{n\to\infty}\|y_n - y_\infty\|_Y = 0$. Then there exist $(u_n)_{n\in\mathbb{N}} \subseteq D(T)$ such that $y_n = Tu_n$ for every $n \in \mathbb{N}$. The fact that $(y_n)_{n\in\mathbb{N}}$ is Cauchy together with the assumed inequality implies that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $X$. Since $(X, \|\cdot\|_X)$ is complete, there exists $u_\infty \in X$ so that $u_n \to u_\infty$ as $n \to \infty$. Since $T$ is closed, $u_n \to u_\infty$ and $Tu_n \to y_\infty$ as $n \to \infty$, it follows that $u_\infty \in D(T)$ and $y_\infty = Tu_\infty \in \operatorname{im}(T)$. Thus, $\operatorname{im}(T)$ is closed.

"$(iii) \Rightarrow (i)$": Since $\operatorname{im}(T)$ is closed and $T$ is a closed operator, Banach's closed range theorem implies that $\operatorname{im}(T^*) = (\ker(T))^\perp$. Since $T$ is in addition injective, $\operatorname{im}(T^*) = X^*$, i.e., $T^*$ is surjective.