13.1. Friedrich extension

Let $(H, \langle\cdot,\cdot\rangle_H)$ be a $\mathbb{K}$-Hilbert space (with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) and let $A\colon D_A \subseteq H \to H$ be a densely defined linear operator satisfying that

- A is *symmetric*, i.e., $\forall x, y \in D_A\colon \langle Ax, y\rangle_H = \langle x, Ay\rangle_H$ and
- A is *bounded below*, i.e., there exists $C \in \mathbb{R}$ such that $\langle Ax, x\rangle_H \geq C\|x\|_H^2$ for all $x \in D_A$.

Our goal is to show that A possesses a self-adjoint extension B (i.e., $A \subseteq B = B^*$) with $\langle Bx, x\rangle_H \geq C\|x\|_H^2$ for all $x \in D_B$.

(a) Find $\lambda \in \mathbb{R}$, $\varepsilon \in (0, \infty)$ so that $a\colon D_A \times D_A \ni (x, y) \mapsto \langle Ax + \lambda x, y\rangle_H \in \mathbb{K}$ defines an inner product on D_A which satisfies for all $x \in D_A$ that $a(x, x) \geq \varepsilon\|x\|_H^2$.

Solution: For any $\varepsilon \in (0, \infty)$ it holds with $\lambda := \varepsilon - C$ that

$$a(x, x) = \langle Ax + \lambda x, x\rangle_H \geq C\|x\|_H^2 + (\varepsilon - C)\|x\|_H^2 = \varepsilon\|x\|^2 \quad \text{for all } x \in D_A.$$

This demonstrates in particular non-negativity and positive-definiteness of a. Moreover, it holds for all $x, y \in D_A$ (by symmetry of A, the properties of $\langle\cdot,\cdot\rangle_H$, and $\lambda \in \mathbb{R}$) that

$$\begin{aligned} a(x, y) &= \langle Ax + \lambda x, y\rangle_H = \langle Ax, y\rangle_H + \lambda\langle x, y\rangle_H = \langle x, Ay\rangle_H + \lambda\langle x, y\rangle_H \\ &= \overline{\langle Ay, x\rangle_H} + \lambda\overline{\langle y, x\rangle_H} = \overline{\langle Ay, x\rangle_H + \lambda\langle y, x\rangle_H} = \overline{\langle Ay + \lambda y, x\rangle_H} = \overline{a(y, x)}. \end{aligned}$$

Furthermore, we clearly have for all $\mu \in \mathbb{K}$, $x, y, z \in D_A$ that $a(\mu x + y, z) = \mu a(x, z) + a(y, z)$.

(b) Consider the metric space (D_A, d_A) where $d_A(x, y) := \sqrt{a(x - y, x - y)}$ for all $x, y \in D_A$ with a as in (a). Let (K, d_K, ι) be a completion of (D_A, d_A) (cp. Problem 3.3 (*Completion of metric spaces*)). Prove that there exists a unique vector space structure on K so that ι is linear and the vector space operations $K \times K \ni (x, y) \mapsto x + y \in K$ and $\mathbb{K} \times K \ni (\mu, x) \mapsto \mu x \in K$ are continuous (w.r.t. the obvious choices of topologies). In addition, show that there even exists a unique scalar product $\langle\cdot,\cdot\rangle_K\colon K \times K \to \mathbb{K}$ such that for all $x, y \in K$ it holds that $d_K(x, y) = \sqrt{\langle x - y, x - y\rangle_K}$.

Solution: Note that, since ι shall be linear and the vector operations on K shall be continuous, the only possible way to define the sum $x_\infty + y_\infty$ and the product μx_∞ for $x_\infty, y_\infty \in K$ and $\mu \in \mathbb{K}$, is via

$$x_\infty + y_\infty = \lim_{n\to\infty} \iota(x_n + y_n) \quad \text{and} \quad \mu x_\infty = \lim_{n\to\infty} \iota(\mu x_n),$$

where limits are to be understood w.r.t. (K, d_K) and where $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subseteq D_A$ are such that $\limsup_{n\to\infty} d_K(\iota(x_n), x_\infty) = 0$ and $\limsup_{n\to\infty} d_K(\iota(y_n), y_\infty) = 0$.

For this to be well-defined, it needs to be checked that these limits always exist and coincide when using other sequences in $\iota(D_A)$ converging to x_∞ and y_∞, respectively. Indeed, if $\mu \in \mathbb{K}$ and $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subseteq D_A$ are sequences satisfying $\limsup_{n\to\infty} d_K(\iota(x_n), x_\infty) = 0$ and $\limsup_{n\to\infty} d_K(\iota(y_n), y_\infty) = 0$, then

$$\begin{aligned}
&\limsup_{N\to\infty} \sup_{m,n\geq N} d_K(\iota(x_n + y_n), \iota(x_m + y_m))\\
&= \limsup_{N\to\infty} \sup_{m,n\geq N} d_A(x_n + y_n, x_m + y_m)\\
&\leq \limsup_{N\to\infty} \sup_{m,n\geq N} (d_A(x_n, x_m) + d_A(y_n, y_m)) = 0
\end{aligned}$$

as well as

$$\limsup_{N\to\infty} \sup_{m,n\geq N} d_A(\mu x_n, \mu x_m) \leq \mu \limsup_{N\to\infty} \sup_{m,n\geq N} d_A(x_n, x_m) = 0$$

i.e., $(\iota(x_n + y_n))_{n\in\mathbb{N}}$ and $(\iota(\mu x_n))_{n\in\mathbb{N}}$ are Cauchy sequences in (K, d_K). Moreover, for sequences $(x_n^{(1)})_{n\in\mathbb{N}}, (x_n^{(2)})_{n\in\mathbb{N}}, (y_n^{(1)})_{n\in\mathbb{N}}, (y_n^{(2)})_{n\in\mathbb{N}} \subseteq D_A$ satisfying for $i \in \{1, 2\}$ that $\iota(x_n^{(i)}) \to x_\infty$ and $\iota(y_n^{(i)}) \to y_\infty$ in (K, d_K) as $n \to \infty$, it holds that

$$\begin{aligned}
&\limsup_{n\to\infty} d_K(\iota(x_n^{(1)} + y_n^{(1)}), \iota(x_n^{(2)} + y_n^{(2)}))\\
&= \limsup_{n\to\infty} d_A(x_n^{(1)} + y_n^{(1)}, x_n^{(2)} + y_n^{(2)})\\
&\leq \limsup_{n\to\infty} (d_A(x_n^{(1)}, x_n^{(2)}) + d_A(y_n^{(1)}, y_n^{(2)})) = 0
\end{aligned}$$

and

$$\begin{aligned}
&\limsup_{n\to\infty} d_K(\iota(\mu x_n^{(1)}), \iota(\mu x_n^{(2)}))\\
&= \limsup_{n\to\infty} d_A(\mu x_n^{(1)}, \mu x_n^{(2)}) = \mu \limsup_{n\to\infty} d_A(x_n^{(1)}, x_n^{(2)}) = 0.
\end{aligned}$$

Similarly, for the scalar product, the only possible definition for $x_\infty, y_\infty \in K$ is

$$\langle x_\infty, y_\infty\rangle_K = \lim_{n\to\infty} a(x_n, y_n),$$

where $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subseteq D_A$ are such that $\limsup_{n\to\infty} d_K(\iota(x_n), x_\infty) = 0$ and $\limsup_{n\to\infty} d_K(\iota(y_n), y_\infty) = 0$. Again, for this to be well-defined, it needs to be checked that these limits always exist and do not depend on the particular choice of approximating sequences in $\iota(D_A)$. Indeed, for any $x_\infty, y_\infty \in K$, and all $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subseteq D_A$ satisfying that $\iota(x_n) \to x_\infty$ and $\iota(y_n) \to y_\infty$ in (K, d_K) as $n \to \infty$, we obtain that

$$\begin{aligned}
&\limsup_{N\to\infty} \sup_{m,n\geq N} |a(x_n, y_n) - a(x_m, y_m)|\\
&\leq \limsup_{N\to\infty} \sup_{m,n\geq N} \Big(|a(x_n, y_n - y_m)| + |a(x_n - x_m, y_m)|\Big)\\
&\leq \limsup_{N\to\infty} \sup_{m,n\geq N} \Big(d_A(x_n, 0) d_A(y_n, y_m) + d_A(y_m, 0) d_A(x_n, x_m)\Big) = 0,
\end{aligned}$$

i.e., that $(a(x_n, y_n))_{n\in\mathbb{N}} \subseteq \mathbb{K}$ is a Cauchy sequence and the limit $\lim_{n\to\infty} a(x_n, y_n)$ exists. Moreover, whenever we have sequences $(x_n^{(1)})_{n\in\mathbb{N}}, (x^{(2)})_{n\in\mathbb{N}}, (y_n^{(1)})_{n\in\mathbb{N}}, (y^{(2)})_{n\in\mathbb{N}} \subseteq D_A$ with $\iota(x_n^{(i)}) \to x_\infty$ and $\iota(y_n^{(i)})_{n\in\mathbb{N}} \to y_\infty$ in (K, d_K) as $n \to \infty$ for $i \in \{1, 2\}$, it holds that

$$\begin{aligned}
&\limsup_{n\to\infty} |a(x_n^{(1)}, y_n^{(1)}) - a(x_n^{(2)}, y_n^{(2)})| \\
&\leq \limsup_{n\to\infty} \Big(|a(x_n^{(1)}, y_n^{(1)} - y_n^{(2)})| + |a(x_n^{(1)} - x_n^{(2)}, y_n^{(2)})| \Big) \\
&\leq \sup_{m\in\mathbb{N}} \sqrt{a(x_m^{(1)}, x_m^{(1)})} \limsup_{n\to\infty} \sqrt{a(y_n^{(1)} - y_n^{(2)}, y_n^{(1)} - y_n^{(2)})} \\
&+ \sup_{m\in\mathbb{N}} \sqrt{a(y_m^{(2)}, y_m^{(2)})} \limsup_{n\to\infty} \sqrt{a(x_n^{(1)} - x_n^{(2)}, x_n^{(1)} - x_n^{(2)})} = 0,
\end{aligned}$$

i.e., $\lim_{n\to\infty} a(x_n^{(1)}, y_n^{(1)}) = \lim_{n\to\infty} a(x_n^{(2)}, y_n^{(2)})$. Thus, there exists a map $\langle\cdot,\cdot\rangle_K \colon K \times K \to \mathbb{K}$ satisfying that

$$\langle x_\infty, y_\infty\rangle_K = \lim_{n\to\infty} a(x_n, y_n)$$

whenever $\limsup_{n\to\infty} d_K(\iota(x_n), x_\infty) = 0 = \limsup_{n\to\infty} d_K(\iota(y_n), y_\infty)$. Moreover, $\langle\cdot,\cdot\rangle_K$ clearly satisfies for all $\mu \in \mathbb{K}$, $x, y, z \in K$ that

$$\langle x, y\rangle_K = \overline{\langle y, x\rangle_K} \quad \text{and} \quad \langle \mu x + y, z\rangle_K = \mu\langle x, z\rangle_K + \langle y, z\rangle_K.$$

In addition, it holds for all $x_\infty \in K$ with $x_\infty = \lim_{n\to\infty} \iota(x_n)$ in K and $(x_n)_{n\in\mathbb{N}} \subseteq D_A$ that

$$\langle x_\infty, x_\infty\rangle_K = \lim_{n\to\infty} a(x_n, x_n) \geq 0$$

and equality on the right hand side would just imply that

$$d_K(x_\infty, \iota(0)) = \lim_{n\to\infty} d_A(x_n, 0) = \lim_{n\to\infty} \sqrt{a(x_n, x_n)} = 0, \text{ i.e., } x_\infty = \iota(0) = 0_K.$$

Finally, note that for all $x_\infty, y_\infty \in K$ and $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subseteq D_A$ with $\iota(x_n) \to x_\infty$ and $\iota(y_n) \to y_\infty$ in K as $n \to \infty$, we get that

$$\begin{aligned}
d_K(x_\infty, y_\infty) &= \lim_{n\to\infty} d_K(\iota(x_n), \iota(y_n)) = \lim_{n\to\infty} d_A(x_n, y_n) \\
&= \lim_{n\to\infty} \sqrt{a(x_n - y_n, x_n - y_n)} = \sqrt{\langle x_\infty - y_\infty, x_\infty - y_\infty\rangle_K}.
\end{aligned}$$

(c) With (K, d_K, ι) being a completion of (D_A, d_A), equipped with the Hilbert space structure (in particular, with the scalar product $\langle\cdot,\cdot\rangle_K \colon K \times K \to \mathbb{K}$) shown to exist in (b), argue that there exists an injective bounded linear map $J\colon K \to H$ satisfying for all $x \in D_A$, $y \in K$ that

$$\langle \iota(x), y\rangle_K = \langle Ax + \lambda x, Jy\rangle_H.$$

Moreover, prove that $J(K) \subseteq H$ can be written as

$$J(K) = \Big\{ x_\infty \in H \mid \exists (x_n)_{n\in\mathbb{N}} \subseteq D_A \text{ with } \limsup_{n\to\infty} \|x_n - x_\infty\|_H = 0 \text{ and } \limsup_{N\to\infty} \sup_{m,n\geq N} a(x_n - x_m, x_n - x_m) = 0 \Big\}.$$

Solution: The linear map $D_A \ni x \mapsto x \in H$ is $\frac{1}{\sqrt{\varepsilon}}$-Lipschitz according to (a):

$$\sqrt{\varepsilon}\|x-y\|_H \leq \sqrt{a(x-y, x-y)} = d_A(x,y) = d_K(x,y) = \|x-y\|_K \quad \text{for all } x, y \in D_A.$$

Using the universal property of completions of metric spaces (cf. Problem 3.3(a)), we obtain that there exists a unique $\frac{1}{\sqrt{\varepsilon}}$-Lipschitz extension $J\colon K \to H$ satisfying that $J(\iota(x)) = x$ for all $x \in D_A$. Clearly, J is linear as

$$\begin{aligned} J(\mu x_\infty + y_\infty) &= \lim_{n\to\infty} J(\mu\iota(x_n) + \iota(y_n)) = \lim_{n\to\infty} J(\iota(\mu x_n + y_n)) = \lim_{n\to\infty} (\mu x_n + y_n) \\ &= \mu \lim_{n\to\infty} J(\iota(x_n)) + \lim_{n\to\infty} J(\iota(y_n)) = \mu J(x_\infty) + J(y_\infty) \end{aligned}$$

for all $\mu \in \mathbb{K}$, $x_\infty, y_\infty \in K$ and $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subseteq D_A$ with $\limsup_{n\to\infty}[\|\iota(x_n) - x_\infty\|_K + \|\iota(y_n) - y_\infty\|_K] = 0$.

Since for all $x, y \in D_A$ it holds that

$$\langle \iota(x), \iota(y)\rangle_K = a(x,y) = \langle Ax + \lambda x, y\rangle_H, = \langle Ax + \lambda x, J(\iota(y))\rangle_H.$$

we obtain by density of $\iota(D_A)$ in K and by continuity of J that

$$\langle \iota(x), y\rangle_K = \langle Ax + \lambda x, Jy\rangle_H$$

for all $x \in D_A$, $y \in K$.

Finally, observe that the characterization of $J(K)$ follows from the fact that J is continuous and injective and that the elements of K just correspond to equivalence classes of Cauchy sequences w.r.t. (D_A, a).

(d) Show that operator $B\colon D_B \subseteq H \to H$, defined by $D_B := \operatorname{im}(JJ^*)$, $B(JJ^*u) := u - \lambda JJ^*u$ for all $u \in H$, is well-defined and a self-adjoint extension of A (i.e., $D_A \subseteq D_B$) with $\langle Bx, x\rangle_H \geq C\|x\|_H^2$ for all $x \in D_B$.

Solution: Since $J\colon K \to H$ is an injective bounded linear map with dense image (as $D_A \subseteq J(K)$), Problem 12.4 (*Special construction of self-adjoint operators*) ensures that $(JJ^*)^{-1}\colon \operatorname{im}(JJ^*) \subseteq H \to H$ is self-adjoint. Hence, $B = (JJ^*)^{-1} - \lambda$ is well-defined and also self-adjoint. Moreover, for all $x \in D_A$, $u \in H$ it holds that

$$\begin{aligned}\langle Ax, JJ^*u\rangle_H &= \langle Ax + \lambda x, JJ^*u\rangle_H - \lambda\langle x, JJ^*u\rangle_H = \langle \iota(x), J^*u\rangle_K - \lambda\langle x, JJ^*u\rangle_H \\ &= \langle J\iota(x), u\rangle_H - \lambda\langle x, JJ^*u\rangle_H = \langle x, u - \lambda JJ^*u\rangle_H = \langle x, B(JJ^*u)\rangle_H.\end{aligned}$$

Hence, it holds for all $x \in D_A$ that $x \in D_{B^*} = D_B$ and $Bx = B^*x = Ax$. Moreover, for all $x \in D_B$, it holds that there exists $w \in H$ with $x = JJ^*w$ and

$$\begin{aligned}\langle Bx, x\rangle_H &= \langle B(JJ^*w), JJ^*w\rangle_H = \langle w - \lambda JJ^*w, JJ^*w\rangle_H \\ &= \|J^*w\|_K^2 - \lambda\|JJ^*w\|_H^2 \geq \varepsilon\|JJ^*w\|_H^2 - \lambda\|JJ^*w\|_H^2 \\ &= C\|JJ^*w\|_H^2 = C\|x\|_H^2,\end{aligned}$$

where we used that $\lambda = C - \varepsilon$ and that J is $\frac{1}{\sqrt{\varepsilon}}$-Lipschitz.

13.2. The Dirichlet-Laplace operator as a Friedrich extension

Let $A\colon C_c^\infty((0,1),\mathbb{R}) \subseteq L^2((0,1),\mathbb{R}) \to L^2((0,1),\mathbb{R})$ be defined by $Af = -f''$ for all $f \in C_c^\infty((0,1),\mathbb{R})$. Our goal is to construct the Friedrich extension of A.

(a) Prove that $(C_c^\infty((0,1),\mathbb{R}), a)$ with a defined via $a(u,v) = \int_{(0,1)} u'v'\,dx$ for all $u, v \in C_c^\infty((0,1),\mathbb{R})$ is an inner product space and prove that there exists $c \in (0,\infty)$ such that for all $u \in C_c^\infty((0,1),\mathbb{R})$ it holds that $\int_0^1 |u|^2\,dx \leq c\int_0^1 |u'|^2\,dx$.

Solution: Clearly, $C_c^\infty((0,1),\mathbb{R})$ (with the usual operations) is an $\mathbb{R}$-vector space. Moreover, it holds for all $u, v, w \in C_c^\infty((0,1),\mathbb{R})$, $\lambda \in \mathbb{R}$ that

$$a(\lambda u + v, w) = \lambda a(u,w) + a(v,w), \quad a(u,v) = a(v,u), \quad \text{and} \quad a(u,u) \geq 0.$$

In addition, for every $u \in C_c^\infty((0,1),\mathbb{R})$ it holds due to the fundamental theorem of calculus that

$$u(x) = \int_0^x u'(t)\,dt \quad \text{for all } x \in (0,1).$$

The Cauchy–Schwarz inequality hence implies for all $u \in C_c^\infty((0,1),\mathbb{R})$ that

$$\begin{aligned}\int_0^1 |u(x)|^2\,dx &\leq \int_0^1 \left(\int_0^x |u'(t)|\,dt\right)^2 dx \leq \int_0^1 \int_0^x |u'(t)|^2\,dt \int_0^x 1\,dt\,dx \\ &\leq \int_0^1 \int_0^1 |u'(t)|^2\,dt\,dx = \int_0^1 |u'(t)|^2\,dt.\end{aligned}$$

In particular, for $u \in C_c^\infty((0,1),\mathbb{R})$, $a(u,u) = 0$ implies that $\|u\|_{L^2((0,1),\mathbb{R})} = 0$, i.e., $u \equiv 0$. This completes the proof that a is an inner product.

(b) Let $K \subseteq L^2((0,1),\mathbb{R})$ be such that $u \in K$ if and only if there exists a sequence $(f_n)_{n\in\mathbb{N}} \subseteq C_c^\infty((0,1),\mathbb{R})$ such that

- $f_n \to u$ in $L^2((0,1),\mathbb{R})$ as $n \to \infty$ and
- $(f_n')_{n\in\mathbb{N}} \subseteq L^2((0,1),\mathbb{R})$ is a Cauchy sequence.

Prove that, for every $u \in K$, there exists a unique $w \in L^2((0,1),\mathbb{R})$ such that

$$\int_{(0,1)} w\varphi\,dx = -\int_{(0,1)} u\varphi'\,dx \quad \text{for all } \varphi \in C_c^\infty((0,1),\mathbb{R}).$$

Afterwards, we shall always write $w = u'$ in such a situation (as w equals the classical derivative in the case of smooth functions).

Solution: Let $u \in K$. Then there exists $(f_n)_{n\in\mathbb{N}} \subseteq C_c^\infty((0,1),\mathbb{R})$ such that $f_n \to u$ in $L^2((0,1),\mathbb{R})$ as $n \to \infty$ and $(f_n')_{n\in\mathbb{N}} \subseteq L^2((0,1),\mathbb{R})$ is a Cauchy sequence. Since $L^2((0,1),\mathbb{R})$ is a Hilbert space, there exists $w \in L^2((0,1),\mathbb{R})$ such that $f_n' \to w$ as $n \to \infty$. Moreover, for all $\varphi \in C_c^\infty((0,1),\mathbb{R})$ it holds that

$$\int_{(0,1)} w\varphi\,dx = \lim_{n\to\infty}\int_{(0,1)} f_n'\varphi\,dx = -\lim_{n\to\infty}\int_{(0,1)} f_n\varphi'\,dx = -\int_{(0,1)} u\varphi'\,dx.$$

Finally, if $(g_n)_{n\in\mathbb{N}} \subseteq C_c^\infty((0,1),\mathbb{R})$ is another sequence satisfying that $g_n \to u$ in $L^2((0,1),\mathbb{R})$ as $n \to \infty$ and $(g_n')_{n\in\mathbb{N}} \subseteq L^2((0,1),\mathbb{R})$ is a Cauchy sequence (with limit $v \in L^2((0,1),\mathbb{R})$), then the previous considerations imply that

$$\int_{(0,1)} (w-v)\varphi\,dx = 0 \quad \text{for all } \varphi \in C_c^\infty((0,1),\mathbb{R}).$$

By the fundamental theorem of the calculus of variations (or by $C_c^\infty((0,1),\mathbb{R})$ being dense in $L^2((0,1),\mathbb{R})$), $w = v$ in $L^2((0,1),\mathbb{R})$.

(c) Prove that $\langle\cdot,\cdot\rangle_K \colon K \times K \ni (u,v) \mapsto \int_{(0,1)} u'v'\,dx \in \mathbb{R}$ defines a scalar product on K and that $(K, \langle\cdot,\cdot\rangle_K)$ is a completion of $(C_c^\infty((0,1),\mathbb{R}), a)$.

Solution: For any sequence $(f_n)_{n\in\mathbb{N}} \subseteq C_c^\infty((0,1),\mathbb{R})$ which is Cauchy w.r.t. a, part (b) implies that $(f_n)_{n\in\mathbb{N}}$ and $(f_n')_{n\in\mathbb{N}}$ are Cauchy sequences in $L^2((0,1),\mathbb{R})$. By part (b) again, there exists $u \in K$ such that $f_n \to u$ and $f_n' \to u'$ in $L^2((0,1),\mathbb{R})$ as $n \to \infty$. Moreover, equivalent Cauchy sequences give rise to the same element of K. Conversely, every element $u \in K$ can be identified with an equivalence class of Cauchy sequences w.r.t. a. Finally, for $u, v \in K$ and $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}} \subseteq C_c^\infty((0,1),\mathbb{R})$ with $f_n \to u$, $f_n' \to u'$, $g_n \to v$ and $g_n' \to v'$ in $L^2((0,1),\mathbb{R})$ as $n \to \infty$, we obtain

$$\langle u,v\rangle_K = \int_{(0,1)} u'v'\,dx = \lim_{n\to\infty}\int_{(0,1)} f_n'g_n'\,dx = \lim_{n\to\infty} a(f_n,g_n).$$

(d) Prove that the Friedrich extension B as in Problem 13.1 (*Friedrich extension*) is given as follows: $u \in D_B$ if and only if $u \in K$ and there exists $g \in L^2((0,1),\mathbb{R})$ such that for all $\varphi \in C_c^\infty((0,1),\mathbb{R})$ it holds that $\int_0^1 u\varphi''\,dx = \int_0^1 g\varphi\,dx$; in this case, $-g = Bu$.

Solution: Note that it holds for all $u, v \in C_c^\infty((0,1),\mathbb{R})$ that

$$a(u,v) = \int_{(0,1)} u'v'\,dx = \int_{(0,1)} -u''v\,dx.$$

By (b), we may carry out the construction in 13.1 with $\lambda = 0$ and $\varepsilon = 1$ and even identify K as well as $J(K)$ (from 13.1) with K in the current context. With these choices, J is just the embedding of K into $L^2((0,1),\mathbb{R})$. Moreover, for every $f \in L^2((0,1),\mathbb{R})$, $u = J^*f \in K$ is the unique element of K satisfying that

$$\langle u, v\rangle_K = \int_{(0,1)} u'v'\,dx = \int_{(0,1)} fJv\,dx = \int_{(0,1)} fv\,dx \quad \text{for all } v \in K$$

and equivalently, by density of $C_c^\infty((0,1),\mathbb{R})$,

$$\int_{(0,1)} u'\varphi'\,dx = \int_{(0,1)} fJ\varphi\,dx = \int_{(0,1)} f\varphi\,dx \quad \text{for all } v \in C_c^\infty((0,1),\mathbb{R}).$$

Thus,

$$\int_{(0,1)} u\varphi''\,dx = -\int_{(0,1)} u'\varphi'\,dx = \int_{(0,1)} -f\varphi\,dx \quad \text{for all } \varphi \in C_c^\infty((0,1),\mathbb{R}).$$

Since it holds that $JJ^*f = Ju = u$ in $L^2((0,1),\mathbb{R})$, the above implies that $u \in D_B$ and $Bu = (JJ^*)^{-1}u = f$. And conversely, if $u \in K$ and there exists $g \in L^2((0,1),\mathbb{R})$ such that $\int_{(0,1)} u\varphi''\,dx = \int_{(0,1)} g\varphi\,dx$ for all $\varphi \in C_c^\infty((0,1),\mathbb{R})$, then the above yields that with $JJ^*g = -u$, i.e., $u \in D_B$ and $Bu = -g$.

(e) Prove that the embedding $J\colon (K, \|\cdot\|_K) \ni f \mapsto f \in (L^2(0,1),\mathbb{R}), \|\cdot\|_{L^2})$ is compact. In addition, prove that every element of K has a unique continuous representative and that this continuous representative extends uniquely to a continuous function on $[0,1]$ vanishing on $\{0,1\}$.

Solution: Let $(u_n)_{n\in\mathbb{N}} \subseteq K$ be a bounded sequence. There exist, by (b), $(f_n)_{n\in\mathbb{N}} \subseteq C_c^\infty((0,1),\mathbb{R})$ satisfying for all $n \in \mathbb{N}$ that

$$\|u_n - f_n\|_{L^2}^2 + \|u_n' - f_n'\|_{L^2}^2 \leq \frac{1}{n^2}.$$

By the fundamental theorem of calculus, it holds for all $x_1, x_2 \in [0,1]$ that

$$\sup_{n\in\mathbb{N}} |f_n(x_1) - f_n(x_2)| = \sup_{n\in\mathbb{N}} \left| \int_{x_1}^{x_2} f_n'(t)\,dt \right| \leq \sup_{n\in\mathbb{N}} \|f_n'\|_{L^2} |x_1 - x_2|^{1/2}.$$

This (keeping in mind that $\lim_{x\searrow 0} f_n(x) = 0 = \lim_{x\nearrow 1} f_n(x)$ for every $n\in\mathbb{N}$), implies that the continuous extensions of the functions f_n, $n\in\mathbb{N}$, to $[0,1]$ are uniformly bounded and equicontinuous on $[0,1]$. The Arzéla–Ascoli theorem hence implies that there exist a sequence $(n_k)_{k\in\mathbb{N}}\subseteq\mathbb{N}$ with $n_k\nearrow\infty$ as $k\to\infty$ and a function $f_\infty\in C([0,1],\mathbb{R})$ with $f_\infty(0)=0=f_\infty(1)$ such that

$$\limsup_{k\to\infty}\left[\sup_{x\in(0,1)}|f_{n_k}(x)-f_\infty(x)|\right]=0.$$

Identifying f_∞ slightly sloppily with its $L^2((0,1),\mathbb{R})$-equivalence class, we obtain that

$$\begin{aligned}&\limsup_{k\to\infty}\|u_{n_k}-f_\infty\|_{L^2((0,1),\mathbb{R})}\\ &\le \limsup_{k\to\infty}\|u_{n_k}-f_{n_k}\|_{L^2((0,1),\mathbb{R})}+\limsup_{k\to\infty}\|f_{n_k}-f_\infty\|_{L^2((0,1),\mathbb{R})}\\ &\le \limsup_{k\to\infty}\frac{1}{n_k}+\limsup_{k\to\infty}\|f_{n_k}-f_\infty\|_{L^\infty((0,1),\mathbb{R})}=0.\end{aligned}$$

Finally, for every $u\in K$, there exists a sequence $(f_n)_{n\in\mathbb{N}}\subseteq C_c^\infty((0,1),\mathbb{R})$ satisfying that $f_n\to u$ and $f_n'\to u'$ in $L^2((0,1),\mathbb{R})$ as $n\to\infty$. In other words, $f_n\to u$ in K as $n\to\infty$. The same reasoning as above implies that there exists $f_\infty\in C([0,1],\mathbb{R})$ such that $\limsup_{n\to\infty}\sup_{x\in(0,1)}|f_n(x)-f_\infty(x)|=0$. Since $f_n\to u$ in $L^2((0,1),\mathbb{R})$, it must hold that $u=f_\infty$ a.e.

Remark. Actually, we proved above that $(K,\|\cdot\|_K)$ embeds compactly into the space $(C([0,1],\mathbb{R}),\|\cdot\|_{\sup})$ and if we had paid more attention, we could have proved that $(K,\|\cdot\|_K)$ embeds continuously into the space of Hölder continuous functions with exponent $\frac12$ and compactly into any Hölder space with exponent strictly less than $\frac12$.

(f) Infer that $B^{-1}\colon L^2((0,1),\mathbb{R})\to L^2((0,1),\mathbb{R})$ is a compact operator.

Solution: From the construction of the Friedrich extension we know that $B^{-1}=JJ^*$. Since J is compact (by (e), JJ^* is also compact.

(g) Determine the spectrum of B as well as an orthonormal basis of $L^2((0,1),\mathbb{R})$ consisting of eigenvectors of B (respectively of B^{-1}).

Solution: From to the spectral theory of compact self-adjoint operators on Hilbert spaces, we know that in this case $\sigma(B^{-1})=\sigma_p(B^{-1})\cup\{0\}$. For $f\in L^2((0,1),\mathbb{R})$, we find $u=B^{-1}f$ as the unique element of K satisfying

$$\int_{(0,1)}u'v'\,dx=\int_{(0,1)}fv\,dx\quad\text{for all } v\in K.$$

Thus, u is an eigenvector of B^{-1} with eigenvalue $\mu\in\mathbb{R}\setminus\{0\}$ if and only if

$$\mu\int_{(0,1)}u'v'\,dx=\int_{(0,1)}uv\,dx\quad\text{for all } v\in K.$$

By (e), we may and will from now on consider u as a continuous function on $[0,1]$ vanishing on $\{0,1\}$. Moreover, for all $\varphi \in C_c^\infty((0,1),\mathbb{R})$ it holds that

$$\begin{aligned}\mu \int_{(0,1)} u'\varphi'\,dx &= \int_{(0,1)} u(x)\varphi(x)\,dx \\ &= \int_{(0,1)} u(x)\int_0^x \varphi'(t)\,dt\,dx = \int_{(0,1)}\int_t^1 u(x)\,dx\,\varphi'(t)\,dt.\end{aligned}$$

This implies that there exists $c \in \mathbb{R}$ such that $\mu u'(t) = c + \int_t^1 u(x)\,dx$ for a.e. $t \in (0,1)$. Since the function $(0,1) \ni t \mapsto c + \int_t^1 u(x)\,dx$ is continuously differentiable (actually, it extends to an element of $C^1([0,1],\mathbb{R})$), we obtain that u' has a continuously differentiable representative and u itself is therefore twice continuously differentiable in the classical sense. Moreover, it holds for all $\varphi \in C_c^\infty((0,1),\mathbb{R})$ that

$$\mu \int_{(0,1)} u''\varphi\,dx = -\mu \int_{(0,1)} u'\varphi'\,dx = -\int_{(0,1)} u\varphi\,dx,$$

which implies that $-\mu u''(x) = u(x)$ for a.e. $x \in (0,1)$. Since we consider continuous representatives whenever possible, this relation remains true for every $x \in [0,1]$. Thus, the eigenfunctions we are looking for are just classical (non-trivial) solutions of the ODE boundary value problem

$$\begin{cases} -\mu u''(x) &= u(x) \quad \text{for all } x \in [0,1], \\ u(0) &= 0, \\ u(1) &= 0. \end{cases}$$

It is well known that such a problem has non-trivial solutions if and only if $\mu = \frac{1}{k^2\pi^2}$ for some $k \in \mathbb{N}$ (in which case $u(t) = \alpha \sin(k\pi t)$ for some $\alpha \in \mathbb{R}$). Hence, we obtain that

$$\sigma(B^{-1}) = \sigma_p(B^{-1}) \cup \{0\} = \left\{\frac{1}{k^2\pi^2} \mid k \in \mathbb{N}\right\} \cup \{0\}$$

and

$$\sigma(B) = \sigma_p(B) = \{k^2\pi^2 \mid k \in \mathbb{N}\}.$$

For every $k \in \mathbb{N}$, there is a one-dimensional space of eigenvectors w.r.t. the eigenvalue $k^2\pi^2$, spanned by (the normalized element) $e_k \in L^2((0,1),\mathbb{R})$ satisfying $e_k(t) = \sqrt{2}\sin(k\pi t)$ for a.e. $t \in (0,1)$.

(h) Express B (and, especially, D_B) and B^{-1} with the help of these eigenvalues and eigenvectors. Can you find a way to define B^s for $s \in (0,\infty)$?

Solution: For all $f \in L^2((0,1),\mathbb{R})$, it holds that

$$B^{-1}f = \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} \langle f, e_n \rangle_{L^2((0,1),\mathbb{R})} e_n = \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} \int_{(0,1)} f(x) \sin(n\pi x)\, dx \sin(n\pi \cdot)$$

Since $D_B = \operatorname{im}(B^{-1})$, we obtain the characterization

$$\begin{aligned} D_B &= \left\{ u = \sum_{n=1}^{\infty} \hat{u}_n e_n \in L^2((0,1),\mathbb{R}) \colon \sum_{n=1}^{\infty} n^2\pi^2 \hat{u}_n e_n \in L^2((0,1),\mathbb{R}) \right\} \\ &= \left\{ u = \sum_{n=1}^{\infty} \hat{u}_n e_n \in L^2((0,1),\mathbb{R}) \colon \sum_{n=1}^{\infty} |n^2\pi^2 \hat{u}_n|^2 < \infty \right\} \\ &= \left\{ u = \sum_{n=1}^{\infty} \hat{u}_n e_n \in L^2((0,1),\mathbb{R}) \colon \sum_{n=1}^{\infty} n^4 |\hat{u}_n|^2 < \infty \right\} \end{aligned}$$

and

$$Bu = \sum_{n=1}^{\infty} n^2\pi^2 \hat{u}_n e_n \quad \text{for all } u = \sum_{n=1}^{\infty} \hat{u}_n e_n \in D_B.$$

For $s \in (0,\infty)$, it is reasonable to define

$$D_{B^s} = \left\{ u = \sum_{n=1}^{\infty} \hat{u}_n e_n \in L^2((0,1),\mathbb{R}) \colon \sum_{n=1}^{\infty} |n^{2s} \hat{u}_n|^2 < \infty \right\}$$

as well as

$$Bu = \sum_{n=1}^{\infty} n^{2s}\pi^{2s} \hat{u}_n e_n \quad \text{for all } u = \sum_{n=1}^{\infty} \hat{u}_n e_n \in D_{B^s}.$$

13.3. Spectral properties of generators of C^0-semigroups

Let $(X, \|\cdot\|_X)$ be a Banach space and let $T = (T_t)_{t\in[0,\infty)} \subseteq L(X)$ be a C^0-semigroup, that is,

- $T_0 = I$,
- $T_{t+s} = T_t T_s$ for all $t, s \in [0,\infty)$, and
- $\limsup_{t\searrow 0} \|T_t x - x\|_X = 0$ for all $x \in X$.

We know that there exist $M \in [1,\infty)$, $\omega \in \mathbb{R}$ such that $\|T_t\|_{L(X)} \leq Me^{\omega t}$ for all $t \in [0,\infty)$. The *generator* of T is the operator $A\colon D_A \subseteq X \to X$ defined by

$$D_A := \left\{ x \in X \mid \lim_{t\searrow 0} \frac{T_t x - x}{t} \text{ exists} \right\} \quad \text{and} \quad Ax := \lim_{t\searrow 0} \frac{T_t x - x}{t} \quad \text{for all } x \in D_A.$$

(a) Prove for all $t \in [0,\infty)$, $x \in D_A$ that $T_t x \in D_A$ and $AT_t x = T_t Ax$.

Solution: For all $t \in [0,\infty)$, $x \in D_A$ it holds that

$$\limsup_{s\searrow 0} \left\| \frac{T_s T_t x - T_t x}{t} - T_t Ax \right\|_X = \limsup_{s\searrow 0} \left\| T_t \Big(\frac{T_s x - x}{t} - Ax \Big) \right\|_X$$
$$\leq \|T_t\|_{L(X)} \limsup_{s\searrow 0} \left\| \frac{T_s x - x}{t} - Ax \right\|_X = 0.$$

Hence, for all $t \in [0,\infty)$, $x \in D_A$ it holds that $T_t x \in D_A$ and $AT_t x = T_t Ax$, as claimed.

(b) Show for all $t \in [0,\infty)$, $x \in X$ that $\int_0^t T_s x\, ds \in D_A$ and $A(\int_0^t T_s x\, ds) = T_t x - x$.

Solution: The claim clearly holds true for $t = 0$, $x \in D_A$. For all $t, h \in (0,\infty)$, $x \in X$ it holds that

$$T_h \int_0^t T_s x\, ds - \int_0^t T_s x\, ds = \int_h^{t+h} T_s x\, ds - \int_0^t T_s x\, ds = \int_t^{t+h} T_s x\, ds - \int_0^h T_s x\, ds.$$

Hence, for all $t \in (0,\infty)$, $x \in X$ it holds that

$$\limsup_{h\searrow 0} \left\| \frac{1}{h} \left[T_h \int_0^t T_s x\, ds - \int_0^t T_s x\, ds \right] - (T_t x - x) \right\|_X$$
$$= \limsup_{h\searrow 0} \left\| \frac{1}{h} \int_t^{t+h} (T_s x - T_t x)\, ds - \frac{1}{h} \int_0^h (T_s x - x)\, ds \right\|_X$$
$$\leq \limsup_{h\searrow 0} \frac{1}{h} \int_t^{t+h} \|T_s x - T_t x\|_X\, ds + \limsup_{h\searrow 0} \frac{1}{h} \int_0^h \|T_s x - x\|_X\, ds = 0,$$

i.e., $\int_0^t T_s x\, ds \in D_A$ and $A(\int_0^t T_s x\, ds) = T_t x - x$.

(c) Show for all $t \in [0,\infty)$, $x \in D_A$ that $\int_0^t T_s Ax\, ds = A \int_0^t T_s x\, ds$.

Solution: The claim is clearly true for $t = 0$, $x \in D_A$. Fix now $t \in (0,\infty)$ and $x \in D_A$. Note that, by (a), it holds for all $h \in (0,1)$, $s \in [0,t]$ that

$$\left\| \frac{T_h T_s x - T_s x}{h} - AT_s x \right\|_X = \left\| T_s \Big(\frac{T_h x - x}{h} - Ax \Big) \right\|_X$$
$$\leq \|T_s\|_{L(X)} \left\| \frac{T_h x - x}{h} - Ax \right\|_X$$
$$\leq \sup_{r\in[0,t]} (Me^{\omega r}) \sup_{r\in(0,1)} \left\| \frac{T_r x - x}{r} - Ax \right\|_X < \infty.$$

Lebesgue's dominated convergence theorem therefore assures that

$$\int_0^t T_s Ax\, ds = \lim_{h\searrow 0} \int_0^t T_s \frac{T_h x - x}{h}\, ds = \lim_{h\searrow 0} \frac{1}{h} \int_t^{t+h} T_s x\, ds - \lim_{h\searrow 0} \frac{1}{h} \int_0^h T_s x\, ds, = T_t x - x.$$

According to part (b), we obtain that $\int_0^t T_s Ax\, ds = A \int_0^t T_s x\, ds$.

(d) Prove for all $x \in D_A$ that the function $u := ([0,\infty) \ni t \mapsto T_t x \in X)$ satisfies that $u \in C^1([0,\infty),(X,\|\cdot\|_X)) \cap C([0,\infty),(D_A,\|\cdot\|_{D_A}))$ and that $u'(t) = Au(t)$ for all $t \in [0,\infty)$.

Solution: Fix $x \in D_A$ and $u := ([0,\infty) \ni t \mapsto T_t x \in X)$. We know that $u \in C([0,\infty),(X,\|\cdot\|_X))$. By (a), we have that $u \in C([0,\infty),(D_A,\|\cdot\|_{D_A}))$ since $AT_t x = T_t Ax$ for all $t \in [0,\infty)$, $x \in D_A$ and $([0,\infty) \ni t \mapsto T_t Ax \in X)$ is continuous. For differentiability, note that parts (b) and (c) show for all $t, s \in [0,\infty)$ that

$$T_t x - T_s x = (T_t x - x) - (T_s x - x) = \int_0^t T_r Ax\, dr - \int_0^s T_r Ax\, dr = \int_s^t T_r Ax\, dr.$$

This (and the fact that $[0,\infty) \ni t \mapsto T_t Ax \in X$ is continuous) implies that $u \in C^1([0,\infty),(X,\|\cdot\|_X))$ and that

$$u'(t) = T_t Ax \quad \text{for all } t \in [0,\infty).$$

Part (a) now yields that $u'(t) = AT_t x = Au(t)$ for every $t \in [0,\infty)$.

(e) Demonstrate that A is densely defined and closed.

Solution: To see that A is densely defined, note that it holds according to (b) for all $t \in (0,\infty)$, $x \in X$ that $\frac{1}{t}\int_0^t T_s x\, ds \in D_A$. Since

$$\limsup_{t \searrow 0} \left\| \frac{1}{t}\int_0^t T_s x\, ds - x \right\|_X \le \limsup_{t \searrow 0} \frac{1}{t}\int_0^t \|T_s x - x\|_X\, ds = 0 \quad \text{for all } x \in X,$$

we conclude that D_A is dense in X. To prove that A is closed, let $(x_n)_{n\in\mathbb{N}} \subseteq D_A$ and $x_\infty, y_\infty \in X$ such that $x_n \to x_\infty$ and $Ax_n \to y_\infty$ in X as $n \to \infty$. By (b) and (c), it holds that

$$T_t x_n - x_n = \int_0^t T_s A x_n\, ds \quad \text{for all } n \in \mathbb{N}, t \in (0,\infty).$$

Since it holds for all $t \in (0,\infty)$ that

$$\begin{aligned}\limsup_{n\to\infty} \|(T_t x_n - x_n) - (T_t x_\infty - x_\infty)\|_X &\le \limsup_{n\to\infty} (\|T_t x_n - T_t x_\infty\|_X + \|x_n - x_\infty\|_X) \\ &\le \limsup_{n\to\infty} (\|T_t\|_{L(X)} + 1)\|x_n - x_\infty\|_X = 0\end{aligned}$$

and

$$\begin{aligned}\limsup_{n\to\infty} \left\| \int_0^t T_s A x_n\, ds - \int_0^t T_s y_\infty\, ds \right\|_X &\le \limsup_{n\to\infty} \int_0^t \|T_s\|_{L(X)} \|Ax_n - y_\infty\|_X\, ds \\ &\le t \sup_{s\in[0,t]} (Me^{\omega s}) \limsup_{n\to\infty} \|Ax_n - y_\infty\|_X = 0,\end{aligned}$$

we obtain that

$$T_t x_\infty - x_\infty = \int_0^t T_s y_\infty \, ds \quad \text{for all } t \in (0,\infty).$$

It follows, dividing by t and letting $t \searrow 0$, that $x_\infty \in D_A$ with $Ax_\infty = y_\infty$. Thus, A is a closed operator.

(f) Prove for all $\lambda \in (\omega,\infty)$ that $R_\lambda \in L(X)$, given by $R_\lambda x = \int_0^\infty e^{-\lambda t} T_t x \, dt$ for all $x \in X$, is well-defined and satisfies

- $R_\lambda(\lambda - A)x = x$ for all $x \in D_A$
- $R_\lambda x \in D_A$ and $(\lambda - A)R_\lambda x = x$ for all $x \in X$.

Solution: Fix $\lambda \in (\omega,\infty)$. For all $x \in X$, it holds that $[0,\infty) \ni t \mapsto e^{-\lambda t} T_t x \in X$ is an exponentially decaying continuous function. Hence, we can for every $x \in X$ make sense of the integral $\int_0^\infty e^{-\lambda t} T_t x \, dt$ as

$$\int_0^\infty e^{-\lambda t} T_t x \, dt = \lim_{\tau \to \infty} \int_0^\tau e^{-\lambda t} T_t x \, dt$$

(where we can construct $\int_0^\tau e^{-\lambda t} T_t x \, dt$ as Riemann integral – as we actually have been doing the whole time in connection with C^0-semigroups). You may view these integrals as Lebesgue–Bochner integrals but since they coincide in our case with the probably simpler Riemann integrals, an excursion into measure and integration theory for vector valued functions is not really necessary for our purposes. Linearity of R_λ is clear. For boundedness, note that

$$\|R_\lambda x\|_X \leq \int_0^\infty e^{-\lambda t} \|T_t x\|_X \, dt \leq \int_0^\infty e^{-\lambda t} M e^{\omega t} \|x\|_X \, dt = \frac{M}{\lambda - \omega} \|x\|_X \quad \text{for all } x \in X.$$

This proves that $R_\lambda \in L(X)$ with $\|R_\lambda\|_{L(X)} \leq \frac{M}{\lambda - \omega}$. Next, note that for all $x \in X$, $h \in (0,\infty)$ it holds that

$$\begin{aligned}
T_h R_\lambda x - R_\lambda x &= T_h \lim_{\tau \to \infty} \int_0^\tau e^{-\lambda t} T_t x \, dt - \lim_{\tau \to \infty} \int_0^\tau e^{-\lambda t} T_t x \, dt \\
&= \lim_{\tau \to \infty} \int_0^\tau e^{-\lambda t} T_{t+h} x \, dt - \lim_{\tau \to \infty} \int_0^\tau e^{-\lambda t} T_t x \, dt \\
&= \lim_{\tau \to \infty} e^{\lambda h} \int_h^{\tau + h} e^{-\lambda t} T_t x \, dt - \lim_{\tau \to \infty} \int_0^\tau e^{-\lambda t} T_t x \, dt \\
&= \lim_{\tau \to \infty} e^{\lambda h} \int_0^{\tau + h} e^{-\lambda t} T_t x \, dt - e^{\lambda h} \int_0^h e^{-\lambda t} T_t x \, dt - \lim_{\tau \to \infty} \int_0^\tau e^{-\lambda t} T_t x \, dt \\
&= (e^{\lambda h} - 1) R_\lambda x - e^{\lambda h} \int_0^h e^{-\lambda t} T_t x \, dt.
\end{aligned}$$

Dividing by h and letting $h \searrow 0$, we obtain for all $x \in X$ that $R_\lambda x \in D_A$ and

$$AR_\lambda x = \lambda R_\lambda x - x,$$

i.e., $(\lambda - A)R_\lambda x = x$. On the other hand, for all $x \in D_A$, we obtain – since $[0,\infty) \ni t \mapsto T_t x \in X$ is continuously differentiable with derivative given by $[0,\infty) \ni t \mapsto T_t Ax \in X$ – via integration by parts that

$$\begin{aligned} R_\lambda Ax &= \lim_{\tau\to\infty} \int_0^\tau e^{-\lambda t} T_t Ax \, dt \\ &= \lim_{\tau\to\infty} \left(\Big[e^{-\lambda t} T_t x \Big]_{t=0}^{t=\tau} - \int_0^\tau -\lambda e^{-\lambda t} T_t x \, dt \right) = -x + \lambda R_\lambda x, \end{aligned}$$

i.e., $R_\lambda(\lambda x - Ax) = x$.

(g) Conclude for all $\lambda \in (\omega, \infty)$ that $\lambda \in \varrho(A)$, i.e., $\lambda - A$ is continuously invertible, with

$$\|(\lambda - A)^{-n}\|_{L(X)} \leq \frac{M}{(\lambda - \omega)^n} \quad \text{for all } n \in \mathbb{N}.$$

Solution: We saw already in (f) that $(\omega, \infty) \subseteq \varrho(A)$ with $R_\lambda = (\lambda - A)^{-1}$ for every $\lambda \in (\omega, \infty)$. It just remains to prove the claimed estimates. For this, we appeal to the fact that $\varrho(A) \ni \lambda \mapsto (\lambda - A)^{-1} \in L(X)$ is actually a smooth (even analytic) function and that, by a general property of resolvents, for all $n \in \mathbb{N}$, $\lambda \in (\omega, \infty)$, it holds that

$$\frac{d^n}{d\lambda^n} R_\lambda = (-1)^n n! R_\lambda^{n+1}.$$

By repeated application of Lebesgue's dominated convergence theorem, we obtain for every $\lambda \in (\omega, \infty)$, $n \in \mathbb{N}$, $x \in X$ that

$$\begin{aligned} \left\| \frac{d^n}{d\lambda^n}(R_\lambda x) \right\|_X &= \left\| \int_0^\infty (-t)^n e^{-\lambda t} T_t x \, dt \right\|_X \leq M \int_0^\infty t^n e^{(\omega - \lambda)t} \|x\|_X \, dt \\ &= \frac{M\|x\|_X}{(\lambda - \omega)^{n+1}} \int_0^\infty e^{-s} s^n \, ds = \frac{M\|x\|_X}{(\lambda - \omega)^{n+1}} \Gamma(n+1) \\ &= \frac{M}{(\lambda - \omega)^{n+1}} n! \|x\|_X. \end{aligned}$$

Therefore, it holds for all $n \in \mathbb{N}$ that

$$\|R_\lambda^{n+1}\| \leq \frac{M}{(\lambda - \omega)^{n+1}}.$$

Since the estimate for R_λ itself was already achieved when checking the boundedness of R_λ in part (f), we are done.

Remark. The Hille–Yosida theorem, in fact, ensures that any densely defined closed operator A on a Banach space X which satisfies for some $M, \omega \in \mathbb{R}$ that $(\omega, \infty) \subseteq \varrho(A)$ and $\|(\lambda - A)^{-n}\|_{L(X)} \leq \frac{M}{(\lambda-\omega)^n}$ for all $\lambda \in (\omega, \infty)$, $n \in \mathbb{N}$, is the generator of a C^0-semigroup.

13.4. A heat semigroup

Let $B\colon D_B \subseteq L^2((0,1),\mathbb{R}) \to L^2((0,1),\mathbb{R})$ denote the self-adjoint extension of $A\colon C_c^\infty((0,1),\mathbb{R}) \subseteq L^2((0,1),\mathbb{R}) \to L^2((0,1),\mathbb{R}))$, given by $Af = -f''$ for all $f \in D_A$, which we constructed in Problem 13.2 (*The Dirichlet–Laplace operator as a Friedrich extension*). In this exercise we dwell on the spectral representation of B obtained in part (h) of Problem 13.2 to construct the associated C^0-semigroup.

(a) Prove that there exists a C^0-semigroup $(T_t)_{t\in[0,\infty)} \subseteq L(L^2((0,1),\mathbb{R}))$ whose generator is $-B$.

Solution: From Problem 13.2, we know that there exists an orthonormal basis $(e_n)_{n\in\mathbb{N}} \subseteq L^2((0,1),\mathbb{R})$ of $L^2((0,1),\mathbb{R})$ such that, for every $n \in \mathbb{N}$, $e_n \in D_B$ and $Be_n = n^2\pi^2 e_n$. Moreover, by Problem 13.2(h), it holds that $D_B = \{u = \sum_{n=1}^\infty \hat{u}_n e_n \in L^2((0,1),\mathbb{R})\colon \sum_{n=1}^\infty n^4 |\hat{u}_n|^2 < \infty\}$. Clearly, for every $t \in [0,\infty)$, the map $T_t\colon L^2((0,1),\mathbb{R}) \to L^2((0,1),\mathbb{R})$, defined by $T_t u = \sum_{n=1}^\infty e^{-n^2\pi^2 t}\hat{u}_n e_n$ for $u = \sum_{n=1}^\infty \hat{u}_n e_n \in L^2((0,1),\mathbb{R})$, is a well-defined bounded linear map. It is readily checked that $T_0 = I$ and that $T_{t+s} = T_t T_s$ for all $t, s \in [0,\infty)$. Moreover, for every $u = \sum_{n=1}^\infty \hat{u}_n e_n \in L^2((0,1),\mathbb{R})$, it holds by Lebesgue's dominated convergence theorem that

$$\limsup_{t\searrow 0} \|T_t u - u\|^2_{L^2((0,1),\mathbb{R})} = \limsup_{t\searrow 0} \left[\sum_{n=1}^\infty (e^{-n^2\pi^2 t} - 1)^2 |\hat{u}_n|^2\right] = 0.$$

Thus, $(T_t)_{t\in[0,\infty)}$ is a C^0-semigroup on $L^2((0,1),\mathbb{R})$. For this, observe that, if $u = \sum_{n=1}^\infty \hat{u}_n e_n \in L^2((0,1),\mathbb{R})$ lies in D_B, then it holds for all $t \in (0,1)$, $n \in \mathbb{N}$ that

$$\left|\frac{e^{-n^2\pi^2 t} - 1}{t} + n^2\pi^2\right| = \frac{1}{t}\int_0^t n^2\pi^2(1 - e^{-n^2\pi^2 s})\, ds \leq n^2\pi^2$$

so that Lebesgue's dominated convergence theorem implies that

$$\limsup_{t\searrow 0} \left\|\frac{T_t u - u}{t} + Bu\right\|^2_{L^2((0,1),\mathbb{R})} = \limsup_{t\searrow 0} \left[\sum_{n=1}^\infty \left|\frac{e^{-n^2\pi^2 t} - 1}{t} + n^2\pi^2\right|^2\right] = 0.$$

Hence, if we denote by G the generator of $(T_t)_{t\in[0,\infty)}$, we have that G is an extension of $-B$, i.e., $D_B \subseteq D_G$ and $Gx = -Bx$ for all $x \in D_B$. On the other hand, if

$u = \sum_{n=1}^{\infty} \hat{u}_n e_n \in L^2((0,1),\mathbb{R})$ lies in D_G, then it holds for every $n \in \mathbb{N}$ that

$$\langle Gu, e_n \rangle_{L^2((0,1),\mathbb{R})} = \lim_{t \searrow 0} \left\langle \frac{T_t u - u}{t}, e_n \right\rangle_{L^2((0,1),\mathbb{R})} = \lim_{t \searrow 0} \frac{e^{-n^2\pi^2} t - 1}{t} = -n^2\pi^2.$$

In particular, since $\sum_{n=1}^{\infty} |\langle Gu, e_n \rangle_{L^2((0,1),\mathbb{R})}|^2 < \infty$ for every $u \in D_G$, we infer that $D_G \subseteq D_B$. This completes the proof that $-B$ is the generator of the C^0-semigroup $(T_t)_{t \in [0,\infty)} \subseteq L^2((0,1),\mathbb{R})$.

(b) Prove for all $t \in (0,\infty)$, $f \in L^2((0,1),\mathbb{R})$ that

- $T_t f \in C^\infty([0,1],\mathbb{R})$ in the sense that there is a (necessarily unique) element of $C^\infty([0,1],\mathbb{R})$ in the L^2-equivalence class $T_t f$ and
- $(T_t f)(0) = 0 = (T_t f)(1)$ in the sense that the unique element of $C^\infty([0,1],\mathbb{R})$ in the L^2-equivalence class $T_t f$ takes on the value 0 on $\{0,1\}$.

Solution: Let $t \in (0,\infty)$ and $f = \sum_{n=1}^{\infty} \hat{f}_n e_n \in L^2((0,1),\mathbb{R})$ be arbitrary but fixed. For every $N \in \mathbb{N}$, let us define the function $F_N \colon [0,1] \to \mathbb{R}$ by $F_N(x) := \sqrt{2} \sum_{n=1}^{N} e^{-n^2\pi^2 t} \hat{f}_n \sin(n\pi x)$ for every $x \in [0,1]$. Clearly, for every $N \in \mathbb{N}$, it holds that the L^2-equivalence class $\sum_{n=1}^{N} e^{-n^2\pi^2 t} \hat{f}_n e_n$ contains a smooth representative which vanishes on $\{0,1\}$, namely the function F_N. For all $m \in \mathbb{N}_0$, $N \in \mathbb{N}$, we have

$$F_N^{(m)}(x) = \sqrt{2} \sum_{n=1}^{N} (n\pi)^m e^{-n^2\pi^2 t} \hat{f}_n \sin^{(m)}(n\pi x) \quad \text{for all } x \in [0,1],$$

where $(\cdot)^{(m)}$ shall be used to denote the m^{th} derivative. From the fact that for all $m \in \mathbb{N}_0$ it holds that

$$\begin{aligned}
&\limsup_{N\to\infty} \left[\sup_{M \geq N} \left\| F_M^{(m)} - F_N^{(m)} \right\|_{C([0,1],\mathbb{R})} \right] \\
&= \limsup_{N\to\infty} \left[\sup_{M \geq N} \sup_{x \in [0,1]} \left| \sqrt{2} \sum_{n=N+1}^{M} (n\pi)^m e^{-n^2\pi^2 t} \hat{f}_n \sin^{(m)}(n\pi x) \right| \right] \\
&\leq \sqrt{2} \limsup_{N\to\infty} \left[\sup_{M \geq N} \sum_{n=N+1}^{M} (n\pi)^m e^{-n^2\pi^2 t} |\hat{f}_n| \right] \\
&\leq \underbrace{\sqrt{2} \left(\sum_{n=1}^{\infty} (n\pi)^{2m} e^{-2n^2\pi^2 t} \right)^{1/2}}_{<\infty \text{ (due to } t>0)} \underbrace{\limsup_{N\to\infty} \left(\sup_{M \geq N} \sum_{n=N+1}^{M} |\hat{f}_n|^2 \right)^{1/2}}_{=0} = 0,
\end{aligned}$$

we can infer that there exist functions $(G_m)_{m \in \mathbb{N}_0} \subseteq C([0,1],\mathbb{R})$ satisfying

$$\limsup_{N\to\infty} \left[\sup_{x \in [0,1]} |(\partial_x F_N)(x) - G_m(x)| \right] = 0.$$

(Note that $G_0(0) = 0 = G_0(1)$ since $F_N(0) = 0 = F_N(1)$ for every $N \in \mathbb{N}$.) The fundamental theorem of calculus and continuity of integrals under uniform convergence allow to show for every $m \in \mathbb{N}$ that $G_m = G_0^{(m)}$, i.e., that G_m is the m^{th} derivative of G_0. Moreover, from the fact that F_N converges uniformly to G_0 on $[0,1]$, we can infer that F_N (strictly speaking, the corresponding $L^2((0,1),\mathbb{R})$-equivalence class) converges to G_0 (its $L^2((0,1),\mathbb{R})$-equivalence class) in $L^2((0,1),\mathbb{R})$ as $N \to \infty$. That is, $G_0 = f$ a.e. and $G_0 \in C^\infty([0,1],\mathbb{R})$ with $G_0(0) = 0 = G_0(1)$.

(c) Prove for all $x \in (0,1)$, $f \in L^2((0,1),\mathbb{R})$ that $(0,\infty) \ni t \mapsto (T_t f)(x) \in \mathbb{R}$ is smooth (where we identify $T_t f \in L^2((0,1),\mathbb{R}))$ for $t \in (0,\infty)$, $f \in L^2((0,1),\mathbb{R})$ with its continuous representative, which exists according to (b)) so that $(T_t f)(x)$ is defined for every $x \in (0,1)$.

Solution: Let $x \in (0,1)$ and $f = \sum_{n=1}^\infty \hat{f}_n e_n \in L^2((0,1),\mathbb{R})$ be arbitrary but fixed. For every $N \in \mathbb{N}$, let us define the function $F_N\colon (0,\infty) \to \mathbb{R}$ by $F_N(t) := \sqrt{2}\sum_{n=1}^N e^{-n^2\pi^2 t}\hat{f}_n \sin(n\pi x)$ for every $t \in (0,\infty)$. Clearly, for every $N \in \mathbb{N}$, it holds that $F_N \in C^\infty((0,1),\mathbb{R})$. Moreover, for all $m \in \mathbb{N}_0$, $N \in \mathbb{N}$, it holds

$$F_N^{(m)}(t) = \sqrt{2}\sum_{n=1}^N (-n^2\pi^2)^m e^{-n^2\pi^2 t}\hat{f}_n \sin(n\pi x) \quad \text{for all } t \in (0,\infty).$$

For every $\varepsilon \in (0,1)$, $m \in \mathbb{N}_0$ it holds that

$$\begin{aligned}
&\limsup_{N\to\infty}\left[\sup_{M\geq N}\left\|F_M^{(m)} - F_N^{(m)}\right\|_{C([\varepsilon,\frac{1}{\varepsilon}],\mathbb{R})}\right] \\
&= \limsup_{N\to\infty}\left[\sup_{M\geq N}\sup_{t\in[\varepsilon,\frac{1}{\varepsilon}]}\left|\sqrt{2}\sum_{n=N+1}^M (-n^2\pi^2)^m e^{-n^2\pi^2 t}\hat{f}_n\sin(n\pi x)\right|\right] \\
&\leq \sqrt{2}\limsup_{N\to\infty}\left[\sup_{M\geq N}\sum_{n=N+1}^M (n\pi)^{2m}e^{-n^2\pi^2\varepsilon}|\hat{f}_n|\right] \\
&\leq \underbrace{\sqrt{2}\left(\sum_{n=1}^\infty (n\pi)^{4m}e^{-2n^2\pi^2\varepsilon}\right)^{1/2}}_{<\infty \text{ (due to } t>0)}\underbrace{\limsup_{N\to\infty}\left(\sup_{M\geq N}\sum_{n=N+1}^M|\hat{f}_n|^2\right)^{1/2}}_{=0} = 0.
\end{aligned}$$

From this, we can infer (in a similar fashion as in (b)) that F_N converges locally uniformly to the function $(0,\infty) \ni t \mapsto \sum_{n=1}^\infty \sqrt{2}e^{-n^2\pi^2 t}\hat{f}_n\sin(n\pi x) \in \mathbb{R}$, that the latter is smooth and that its derivatives emerge as a locally uniform limits of the derivatives of F_N.

(d) Prove for all $f \in L^2((0,1),\mathbb{R})$ that $u\colon (0,\infty)\times[0,1] \to \mathbb{R}$, given by $u(t,x) =$

$(T_t f)(x) \in \mathbb{R}$ for all $t \in (0,\infty)$, $x \in [0,1]$, satisfies that

$$\begin{cases} (\partial_t u)(t,x) &= (\partial_x^2 u)(t,x), & \text{for all } t \in (0,\infty), x \in (0,1), \\ u(t,0) &= 0, & \text{for all } t \in (0,\infty), \\ u(t,1) &= 0, & \text{for all } t \in (0,\infty), \\ \limsup_{t\searrow 0} \|u(t,\cdot) - f\|_{L^2} &= 0. \end{cases}$$

Solution: By part (b), we know already that $u(t,0) = 0 = u(t,1)$ for $t \in (0,\infty)$. By part (a), we know that $\limsup_{t\searrow 0}\|u(t,\cdot) - f\|_{L^2((0,1),\mathbb{R})} = 0$. Moreover, $u(t,x) = \sum_{n=1}^\infty e^{-n^2\pi^2 t}\hat{f}_n\sqrt{2}\sin(n\pi x)$ for all $(t,x) \in (0,\infty)\times[0,1]$. Also, we know from (b) and (c), that $[0,1] \ni x \mapsto u(t,x) \in \mathbb{R}$ belongs to $C^\infty([0,1],\mathbb{R})$ for all $t \in (0,\infty)$ and that $(0,\infty) \ni t \mapsto u(t,x) \in \mathbb{R}$ belongs to $C^\infty((0,\infty),\mathbb{R})$ for all $x \in (0,1)$. This time, we finally consider u as a function of t and x, but the overall philosophy stays the same. Defining, for every $N \in \mathbb{N}$, the function $U_N\colon (0,\infty)\times[0,1] \to \mathbb{R}$ via $U_N(t,x) = \sum_{n=1}^N e^{-n^2\pi^2 t}\hat{f}_n\sqrt{2}\sin(n\pi x)$ for all $(t,x) \in (0,\infty)\times[0,1]$, we have for every $N \in \mathbb{N}$ that $U_N \in C^\infty((0,\infty)\times[0,1],\mathbb{R})$ with

$$(\partial_x^k \partial_t^m U_N)(t,x) = \sqrt{2}\sum_{n=1}^N (-n^2\pi^2)^m e^{-n^2\pi^2 t}\hat{f}_n (n\pi)^k \sin^{(k)}(n\pi x)$$

for all $k,m \in \mathbb{N}_0$, $t \in (0,\infty)$, $x \in (0,1)$. (Here, we denoted by $\sin^{(k)}$ the k^{th} derivative of the sine function, which is one of $\pm\sin$ or $\pm\cos$.) For all $m,k \in \mathbb{N}_0$ and every $\varepsilon \in (0,\infty)$, it holds that

$$\begin{aligned}
&\limsup_{N\to\infty}\sup_{M\geq N}\|(\partial_x^k\partial_t^m U_M)(t,x) - (\partial_x^k\partial_t^m U_N)(t,x)\|_{C([\varepsilon,\frac{1}{\varepsilon}]\times[0,1],\mathbb{R})} \\
&= \limsup_{N\to\infty}\left[\sup_{M\geq N}\sup_{(t,x)\in[\varepsilon,\frac{1}{\varepsilon}]\times[0,1]}\left\|\sqrt{2}\sum_{n=N+1}^M (-n^2\pi^2)^m e^{-n^2\pi^2 t}\hat{f}_n(n\pi)^k\sin^{(k)}(n\pi x)\right\|\right] \\
&\leq \limsup_{N\to\infty}\left[\sup_{M\geq N}\left(\sqrt{2}\sum_{n=N+1}^M (n\pi)^{2m+k}e^{-n^2\pi^2\varepsilon}|\hat{f}_n|\right)\right] \\
&\leq \limsup_{N\to\infty}\sup_{M\geq N}\left(2\sum_{n=N+1}^M (n\pi)^{4m+2k}e^{-2n^2\pi^2\varepsilon}\right)^{1/2}\left(\sum_{n=N+1}^M |\hat{f}_n|^2\right)^{1/2} \\
&\leq \underbrace{\left(2\sum_{n=1}^\infty (n\pi)^{4m+2k}e^{-2n^2\pi^2\varepsilon}\right)^{1/2}}_{<\infty \text{ (by } \varepsilon>0)}\underbrace{\limsup_{N\to\infty}\sup_{M\geq N}\left(\sum_{n=N+1}^M|\hat{f}_n|^2\right)^{1/2}}_{=0} = 0.
\end{aligned}$$

Hence, there exist functions $(V_{m,k})_{(m,k)\in\mathbb{N}_0\times\mathbb{N}_0} \subseteq C((0,\infty)\times[0,1],\mathbb{R})$ satisfying for all $k,m \in \mathbb{N}$, $\varepsilon \in (0,1)$ that

$$\limsup_{N\to\infty}\|V_{m,k} - (\partial_t^m\partial_x^k U_N)\|_{C([\varepsilon,\frac{1}{\varepsilon}]\times[0,1],\mathbb{R})} = 0.$$

Leveraging the fundamental theorem of calculus, we obtain that $V_{m,k} = \partial_t^m \partial_x^k V_{0,0}$ for all $m, k \in \mathbb{N}_0$. Also, it easily follows that $V_{0,0} = u$. Moreover, we have that

$$(\partial_t u)(t,x) = \sqrt{2} \sum_{n=1}^{\infty} -n^2\pi^2 e^{-n^2\pi^2 t} \hat{f}_n \sin(n\pi x) \quad \text{for all } (t,x) \in (0,\infty) \times [0,1]$$

and

$$(\partial_x^2 u)(t,x) = \sqrt{2} \sum_{n=1}^{\infty} -n^2\pi^2 e^{-n^2\pi^2 t} \hat{f}_n \sin(n\pi x) \quad \text{for all } (t,x) \in (0,\infty) \times [0,1],$$

which proves that $(\partial_t u)(t,x) = (\partial_x^2 u)(t,x)$ for all $(t,x) \in (0,\infty) \times [0,1]$.

(e) Finally, prove for all $f \in L^2((0,1),\mathbb{R})$, $v \in C^\infty((0,\infty) \times [0,1],\mathbb{R})$ satisfying

$$\begin{cases} (\partial_t v)(t,x) &= (\partial_x^2 v)(t,x) & \text{for all } t \in (0,\infty), x \in (0,1), \\ v(t,0) &= 0 & \text{for all } t \in (0,\infty), \\ v(t,1) &= 0 & \text{for all } t \in (0,\infty), \\ \limsup_{t\searrow 0} \|v(t,\cdot) - f\|_{L^2} &= 0, \end{cases}$$

that $v(t,x) = (T_t f)(x)$ for all $(t,x) \in (0,\infty) \times [0,1]$.

Solution: Let $w\colon (0,\infty) \times [0,1] \to \mathbb{R}$ be defined by $w(t,x) = u(t,x) - v(t,x)$ for all $(t,x) \in (0,\infty) \times [0,1]$, where we reuse the function u introduced in part (d). Since, by part (d) and by assumption, $u, v \in C^\infty((0,\infty) \times [0,1],\mathbb{R})$, it follows that also $w \in C^\infty((0,\infty) \times [0,1],\mathbb{R})$. Therefore, the function $(0,\infty) \ni t \mapsto \int_{(0,1)} |w(t,x)|^2 \, dx \in \mathbb{R}$ is differentiable and it holds for all $t \in (0,\infty)$ that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left[\int_{(0,1)} |w(t,x)|^2 \, dx\right] &= \int_{(0,1)} (\partial_t w)(t,x) w(t,x) \, dx \\ &= \int_{(0,1)} (\partial_x^2 w)(t,x) w(t,x) \, dx \\ &= -\int_{(0,1)} |(\partial_x w)(t,x)|^2 \, dx \leq 0. \end{aligned}$$

This monotonicity property and the assumption on the initial conditions, it follows for all $t \in (0,\infty)$ that

$$\begin{aligned} \int_{(0,1)} |w(t,x)|^2 \, dx &\leq \limsup_{s\searrow 0} \int_{(0,1)} |w(s,x)|^2 \, dx \\ &= \limsup_{s\searrow 0} \|u(s,\cdot) - f + f - v(s,\cdot)\|_{L^2}^2 \\ &\leq 2 \limsup_{s\searrow 0} \|u(s,\cdot) - f\|_{L^2}^2 + 2 \limsup_{s\searrow 0} \|f - v(s,\cdot)\|_{L^2}^2 = 0. \end{aligned}$$

Since w is continuous, this implies that $w(t,x) = 0$ (i.e., $v(t,x) = u(t,x)$) for all $(t,x) \in (0,\infty) \times [0,1]$.