Differential forms (and Stoke's thm) (c.f. Chapter 11)

$$\boxed{\int_a^b f' = f(b) - f(a)}$$

$\boxed{\Lambda_s(\mathbb{R}^{n*})} :=$ vector space of alternating s-linear maps

$(0 \leq s \leq n)$

$$\underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{s} \longrightarrow \mathbb{R}$$

$\Lambda_0(\mathbb{R}^{n*}) := \mathbb{R}$

$\Lambda_s(\mathbb{R}^{n*}) = 0 \quad s \geq n+1$

$f \in \Lambda_s(\mathbb{R}^n)$

- $f(\xi_1, \ldots, \xi_i + \eta, \ldots, \xi_s) = f(\xi_1, \ldots, \xi_i, \ldots \xi_s) + f(\xi_1, \ldots, \eta, \ldots \xi_s)$
- $f(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(s)}) = \operatorname{sgn}(\sigma) f(\xi_1, \ldots \xi_s)$ σ permutation

Exterior product (wedge)

$\alpha \in \Lambda^s(\mathbb{R}^{n*})$, $\beta \in \Lambda^t(\mathbb{R}^{n*})$

$\alpha \wedge \beta \in \Lambda^{s+t}(\mathbb{R}^{n*})$

$$(\alpha \wedge \beta)(\xi_1, \dots, \xi_{s+t}) := \sum_{\sigma \in S_{s,t}} \operatorname{sgn}(\sigma)\, \alpha(\xi_{\sigma(1)}, \dots, \xi_{\sigma(s)})\, \beta(\xi_{\sigma(s+1)}, \dots, \xi_{\sigma(s+t)})$$

$$S_{s,t} = \{\sigma \in S_{s+t} \mid \sigma(1) < \dots < \sigma(s),\ \sigma(s+1) < \dots < \sigma(s+t)\}$$

(s,t)-shuffles

Properties:

- $\wedge$ bilinear
- $a \in \Lambda_0(\mathbb{R}^{n*}) \cong \mathbb{R}$ $\quad a \wedge \alpha = a\alpha$
- $\alpha \wedge \beta = (-1)^{st} \beta \wedge \alpha$
- $(\alpha \wedge \beta) \wedge \gamma = \alpha(\beta \wedge \gamma)$

$e_1, \dots, e_n$ conical basis of $\mathbb{R}^n \rightsquigarrow (e_1^*, e_2^*, \dots, e_n^*)$ dual basis (standard notation)

OUR NOTATION:

$e^1, \dots, e^n$ dual basis of $\Lambda^1(\mathbb{R}^{n*})$ $\quad e^i(e_j) = \delta^i_j$

$\alpha \in \Lambda_s(\mathbb{R}^{n*})$ has the repr.

$$\alpha \overset{(*)}{=} \sum_{1 \le i_1 < \dots < i_s \le n} \alpha_{i_1, \dots, i_s} e^{i_1} \wedge \dots \wedge e^{i_s}$$

$e^{i_\alpha}(\xi_\beta)$

$$e^{i_1} \wedge \dots \wedge e^{i_s} (\xi_1, \dots \xi_s) = \det [\xi_\beta^{i_\alpha}]_{1 \le \alpha, \beta \le s}$$

Def'n A differential form ω of degree s on $U \subset \mathbb{R}^n$ open is a map $U \longrightarrow \Lambda_s(\mathbb{R}^{n*})$ such that given any $(\xi_1, \dots \xi_s)$ vectors of $\mathbb{R}^n$

$x \mapsto \omega_x(\xi_1, \dots, \xi_s)$ is smooth.

→ If $\xi_1, \dots, \xi_s \in C^\infty(U; \mathbb{R}^n)$

$$\omega(\xi_1, \dots, \xi_s) : U \longrightarrow \mathbb{R}$$

$$\omega_x(\xi_1(x), \dots, \xi_s(x))$$

<u>Differential of a scalar fc'n as a diff form of deg 1</u>

$\xi \in \mathbb{R}^n$ $\qquad$ df is a 1-diff form $\qquad$ $f : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$, smooth

given by $df_x(\xi)$ $\qquad$ $\left(df_x : \mathbb{R}^n \to \mathbb{R} \text{ linear} \right)$

$f = x^i : \mathbb{R}^n \to \mathbb{R}$ $\qquad$ hence $df_x \in \Lambda_1(\mathbb{R}^n)^*$

$$\boxed{dx^i = e^i} \iff dx^i(e_j) = \delta^i_j$$

We denote by $\Omega^s(U)$ the space of diff forms of deg s on U

$U \subseteq \mathbb{R}^n$ open

Notice $\forall \omega \in \Omega^s(U)$

$$\omega = \sum_{1 \le i_1 < \dots < i_s \le n} \underbrace{\omega_{i_1 \dots i_s}}_{\in C^\infty(U)} \underbrace{dx^{i_1} \wedge \dots \wedge dx^{i_s}}_{dx^I}$$

(briefly)

$$= \underbrace{\sum_I \omega_I \, dx^I}$$

$I = (i_1, \dots, i_s) \quad i_1 < \dots < i_s$

As in $(*)$, $\quad \omega_{i_1 \dots i_s} = \omega(e_{i_1}, \dots, e_{i_s})$

Theorem (exterior derivative)

$\exists$ unique seq. of linear operators $d : \Omega^s(U) \to \Omega^{s+1}(U)$, $U \overset{\text{open}}{\subset} \mathbb{R}^n$

$s = 0, 1, 2, \dots$, with the following properties:

(1) for $f \in \Omega^0(U) = C^\infty(U)$ $\quad df$ is the usual differential

(2) $d \circ d = 0$

(3) $d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^s \omega \wedge d\theta$

whenever $\omega \in \Omega^s(U)$, $\theta \in \Omega^t(U)$

(4) $d(\omega|_V) = (d\omega)|_V \quad \forall V \subset U$ open

---

$C^\infty(U) = \Omega^0(U)$

proof

uniqueness

$$\omega = \sum_I \overset{\in}{\omega_I} \, dx^I = \sum_I \omega_I \wedge dx^I$$

$$(*) \quad d\omega \overset{(1)+(3)}{=} \sum_I d\omega_I \wedge dx^I + (-1)^0 \omega_I \wedge d(dx^I) \overset{(2)}{=} \sum_I d\omega_I \wedge dx^I$$

$$= \sum_{1 \le i_1, \dots, i_s \le n} \underbrace{d\omega_{i_1, \dots, i_s}} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_s}$$

<u>existence</u> Define d according to (1) and (*)

[Note that by (1) $\quad df(\xi) = \frac{\partial f}{\partial x^i} \xi^i \qquad f \in C^\infty(U) \cong \Omega^0(U)$

$$= \frac{\partial f}{\partial x^i} dx^i(\xi)$$

$$\boxed{df = \frac{\partial f}{\partial x^i} dx^i}$$

(<u>Exercise</u> compute for instance d of $\sin(x_1 x_2)\, dx_1 \wedge dx_2$)

Let us show (2), (3), (4)

(2): suffices $w = f\,dx^I$ on U

$$dw \overset{(*)}{=} df \wedge dx^I = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i \wedge dx^I$$

$$d(dw) = \sum_{i,j} \frac{\partial^2 f}{\partial x^j \partial x^i} \underbrace{dx^j \wedge dx^i}_{-\frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \wedge dx^j} \wedge dx^I = 0$$

summing over all $1 \leq i \leq j \leq n$

(3) $w = f\,\underbrace{dx^I}_{\text{degrees}}$, $\theta = g\,dx^J$

$$w \wedge \theta = f \wedge dx^I \wedge g \wedge dx^J = fg\, dx^I \wedge dx^J$$

$$d(w \wedge \theta) \overset{(*)}{=} d(fg) \wedge dx^I \wedge dx^J$$

by (1) $\boxed{d(fg) = df\,g + f\,dg}$

$$= g\,\underbrace{df \wedge dx^I}_{dw} \wedge dx^J + f\,dg \wedge dx^I \wedge dx^J$$

$d\theta$

$$= d\omega \wedge \theta + (-1)^s \omega \wedge d\theta$$

(4) $x \in U$ $\quad d\omega_x$ $\quad$ Choose $h \in C_c^\infty(U)$ $\quad h \equiv 1$ in a nbhood of x

$$d(h\omega)_x \overset{(3)}{=} \underbrace{(dh)_x}_{0} \wedge \omega_x + \underset{1}{\underset{\shortparallel}{h(x)}}\, d\omega_x$$

∎

$F : U \subset \mathbb{R}^n \longrightarrow V \overset{\text{open}}{\subset} \mathbb{R}^m$ $\quad C^\infty$ $\quad \omega \in \Omega^s(V)$

We define the <u>pull-back form</u> $F^*\omega \in \Omega^s(U)$

$$(F^*\omega)_x(\xi_1, \dots, \xi_s) = \omega_{F(x)}\big(dF_x(\xi_1), \dots, dF_x(\xi_s)\big)$$

Proposition $F: U \to V$ C^∞ (as before) [cf. Prop 11.4 in the notes]

$\omega \in \Omega^s(V)$, $\theta \in \Omega^t(V)$ Then:

(1) $F^*(\omega \wedge \theta) = F^*\omega \wedge F^*\theta$

(2) $F^*(d\omega) = d(F^*\omega)$

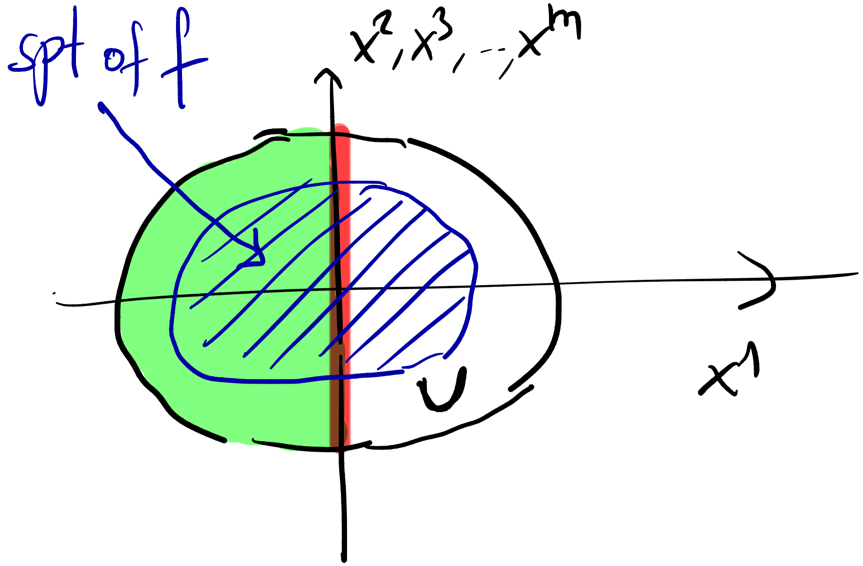
(0) $F^*(\omega + \theta) = F^*\omega + F^*\theta$

proof Exercise

Hint for (2) prove it first for $\omega = f \in C^\infty(V) = \Omega^0(V)$
($\rightsquigarrow$ chain rule)

and the induction over s

Integration of forms and Stokes thm



Baby version of Stokes' thm

$U \subset \mathbb{R}^m$ open $\quad f \in C_c^\infty(U)$

$$\int_{U \cap \{x_1 < 0\}} \frac{\partial f}{\partial x^1} \, dx^1 \dots dx^m = \int_{U \cap \{x_1 = 0\}} f \, dx^1 \dots dx^{m-1}$$

proof

(extend, if you want, f by 0 outside its spt)

$$\text{LHS} = \int_{\mathbb{R}^{m-1} \times (0,\infty)} \frac{\partial f}{\partial x^1} \, dx^1 \dots dx^m \underset{\text{Fubini}}{=} \int_{\mathbb{R}^{m-1}} dx^2 \dots dx^m \int_{-\infty}^{0} \frac{\partial f}{\partial x^1} \, dx^1$$

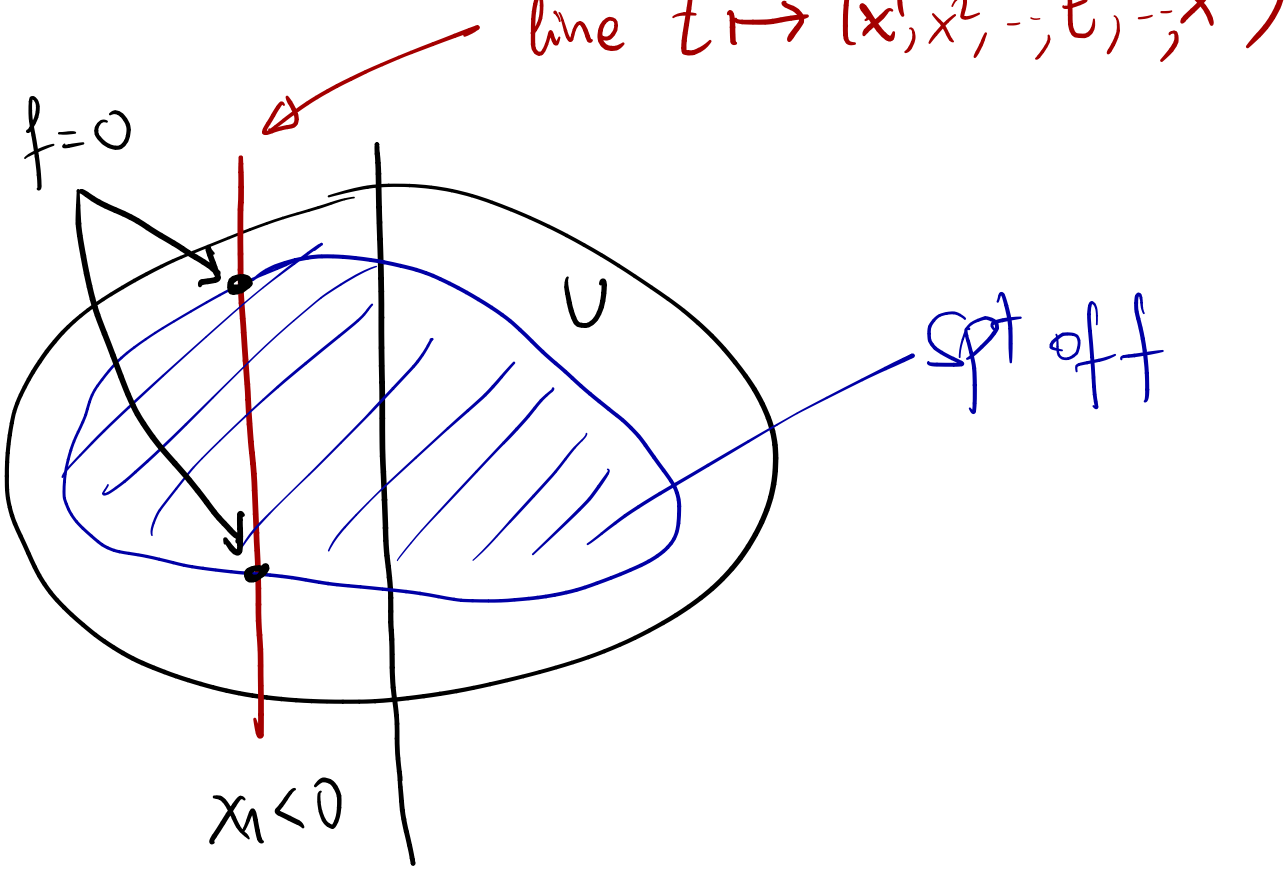
very large value of x_n

$$= \int_{\mathbb{R}^{m-1}} dx^2 \dots dx^m \left[f(\cdot, x^2, \dots, x^m) \right]_{-\infty}^{0} = \int_{\mathbb{R}^{m-1}} f(0, x^2, \dots, x^n) \, dx^2 \dots dx^k$$

$\blacksquare$

Remark similarly for $i = 2, \dots, m$ ($f \in C_c^\infty(U)$ as before)

$$\int_{U \cap \{x_1 < 0\}} \frac{\partial f}{\partial x^i} \, dx^1 \cdots dx^m = 0$$



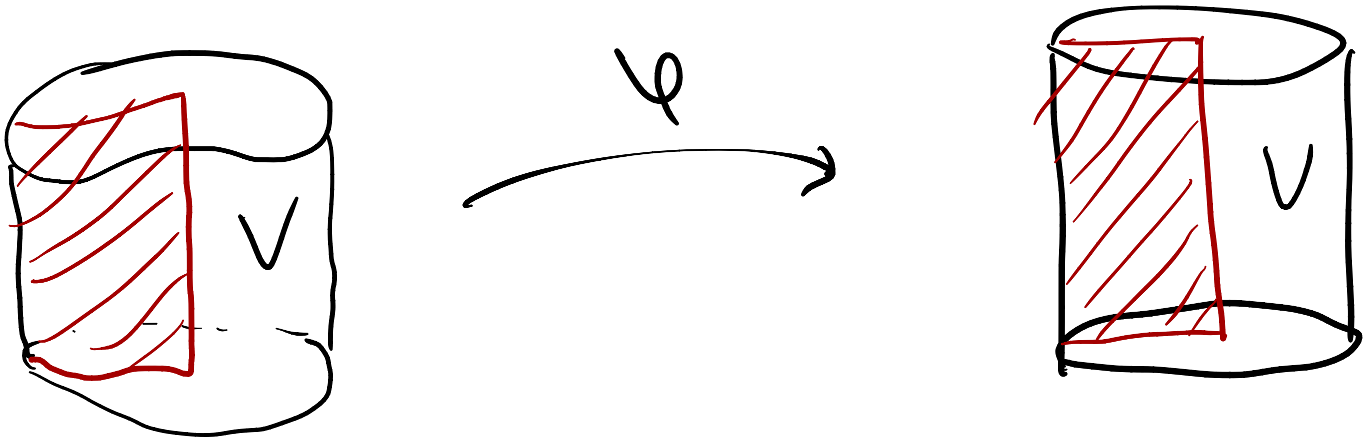
Indeed, by Fubini

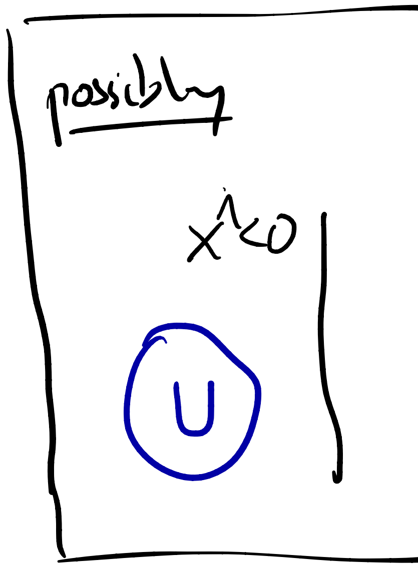
$$\int_{\{x^1 < 0\}} dx^1 dx^{i-1} dx^{i+1} \cdots dx^m \underbrace{\int_{-\infty}^{\infty} \frac{\partial}{\partial x^i} f}_{\text{equals } 0}$$

<u>Def'n</u> A set $M \subset \mathbb{R}^n$ is a m-dim. <u>orientable</u> <u>submfld</u> <u>with bdry</u> of $\mathbb{R}^n$ (of class C^k) if $\forall p \in M$, $\exists$ open nbhd $V \subset \mathbb{R}^n$ of p and a <u>positive</u> C^k-diffeomorphism $\varphi : V \to U$ onto an open set $U \subset \mathbb{R}^n$ st

$\det(d\varphi_q) > 0$ $\forall q \in V$

$$\varphi(M \cap V) = \underbrace{(\mathbb{R}^m \times \{\overset{(0,0,\dots,0)}{\underset{n-m}{0}}\}) \cap U \cap \{x^1 \leq 0\}}$$





$$\partial M := \{p \in M : \varphi(p) \in \{x^1 = 0\} \left[\cap (\mathbb{R}^m \times \{0\})\right]\}$$

(possibly empty)

Def'n Let $\omega \in \Omega^m(\mathbb{R}^n)$ an $M \subset \mathbb{R}^n$ C^∞ m-dim orientable submfld (possibly with ∂). We say the ω is integrable over M if $\exists\ (V_\alpha, \varphi_\alpha)$ submanifold "atlas" (i.e. $\bigcup_\alpha V_\alpha \supset M$) and λ_α partition of unity subordinated to $\{V_\alpha\}$ st.

$$\sum_\alpha \int_{\underbrace{(\mathbb{R}^m \times \{0\}) \cap U_\alpha \cap \{x_m = 0\}}_{\text{subset of } \mathbb{R}^m \times \{0\}} =: W_\alpha \times \{0\}} \left| (\varphi_\alpha^{-1})^*(\lambda_\alpha \omega)(e_1, \dots, e_m) \right| dx^1 \cdots dx^m < \infty$$

If ω is integrable over M, then:

$$\boxed{\int_M \omega := \sum_\alpha \int_{W_\alpha \times \{0\}} (\varphi_\alpha^{-1})^*(\lambda_\alpha \omega)(e_1, \dots, e_m)\, dx^1 \cdots dx^m}$$

By def'n of pull-back of diff forms $\quad (\varphi^{-1} = \tilde{\varphi}^{-1} \circ \psi)$

$$\Big((\varphi^{-1})^* \omega\Big)_{\underbrace{(\bar{x},0)}_{x}} (e_1, \dots, e_n) = \Big((\tilde{\varphi}^{-1})^* \omega\Big)_{\underbrace{(\hat{\varphi}(x'),0)}_{\psi(x)}} \overbrace{\big(d\psi_{(\bar{x},0)}(e_1), \dots, d\psi_{(\bar{x},0)}(e_m)\big)}^{(d\hat{\varphi}_{\bar{x}}(e_1), \dots, d\hat{\varphi}_{\bar{x}}(e_m))}$$

$$= \det(d\hat{\varphi}_{\bar{x}}) \Big((\tilde{\varphi}^{-1})^* \omega\Big)_{(\hat{\varphi}(\bar{x}),0)} (e_1, \dots, e_m)$$

So,

$$\int_W \Big((\varphi^{-1})^* \omega\Big)_{(\bar{x},0)} (e_1, \dots, e_m)\, d\bar{x} = \int_{\hat{\varphi}(W) = \tilde{W}} \det(d\hat{\varphi}_x) \underbrace{\Big((\tilde{\varphi}^{-1})^* \omega\Big)_{(\hat{\varphi}(\bar{x}),0)} (e_1, \dots, e_m)}_{f(\hat{\varphi}(\bar{x}))} \overbrace{d\bar{x}}^{d\bar{y}}$$

$$\left[f(\bar{y}) = (\tilde{\varphi}^{-1})^* \omega_{(\bar{y},0)} (e_1, \dots, e_m) \right] \qquad = \int_{\tilde{W}} \Big((\tilde{\varphi}^{-1})^* \omega\Big)_{(\bar{y},0)} (e_1, \dots, e_m)\, d\bar{y}$$

$$\left[\bar{y} = \hat{\varphi}(\bar{x}) \right]$$

? $\Rightarrow$ ✓

2. (exercise) $\int_M w$ is independent of choice $(V_\alpha, \varphi_\alpha)$, λ_α

Thm (generalized Stokes) $M \subset \mathbb{R}^n$ orientable m-dim submfd (with ∂)

$w \in \Omega^{m-1}(M)$ s.t. $\left\{ \begin{array}{l} w \text{ integrable on } \partial M \\ dw \text{ integrable on } M \end{array} \right.$

Then,

$$\boxed{\int_M dw = \int_{\partial M} w}$$

Orientation of ∂M : $p \in \partial M$ $p \in V$ $\varphi : V \to U$

$\varphi(p) = (0, x^2, x^3, \ldots, x^m, \underbrace{0}_{n-m})$ $\left\{ d(\varphi^{-1})_{\varphi(p)} (e_i) : 2 \leq i \leq m \right\}$

positive basis of $T(\partial M)_p$

proof $\boxed{\text{case 1}}$ w cpt spt in V $\varphi : V \longrightarrow U$

(submanifold chart)

$$\int_M dw = \int_{\mathbb{R}^m \times \{0\} \cap \{x^1 \leq 0\} (\cap U)} (\varphi^{-1})^* dw \, (e_1, \ldots e_m) \, dx_1, \ldots dx_m$$

$$= \int_{\mathbb{R}^m \times \{0\} \cap \{x^1 \leq 0\}} d \underbrace{(\varphi^{-1})^* w}_{\bar{w}} (e_1, \ldots e_m) \, dx_1, \ldots, dx_m$$

$\bar{w}$ is a $(m-1)$-form in U

$J = (j_1, \ldots, j_m)$

$j_1 < j_2 < \ldots < j_m$

$$\bar{w} = \sum_I \bar{w}_I \, dx^I \quad (I = (i_1, \ldots i_{m-1}))$$

$$d\bar{w} = \sum_I \frac{\partial}{\partial x^i} \bar{w}_I \, dx^i \wedge dx^I$$

Observation

$$dx^J(e_1, \ldots e_m) = \begin{cases} 1 & (j_1, \ldots j_m) = (1, \ldots, m) \\ 0 & \text{otherwise} \end{cases}$$

$$d\bar{w} = \sum_i \frac{\partial}{\partial x^i} f^i \underbrace{dx^1\wedge\cdots\wedge dx^m} + \text{other}$$

giving 0 when "contracted" with $(e_1, \dots, e_m)$

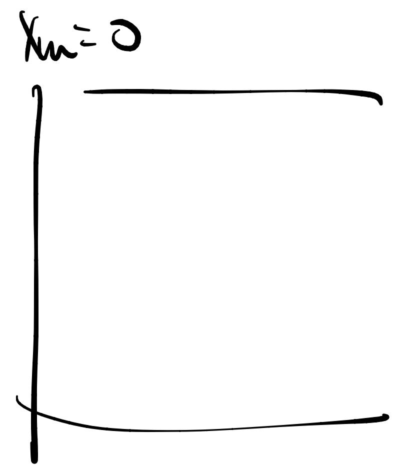
$$\left[\ \bar{w} = \sum_{i=1}^{m} \underbrace{(-1)^{i-1} f^i}\, dx^1\wedge\cdots\wedge \overset{\text{out!}}{\widehat{dx^i}} \wedge\cdots\wedge dx^m + \text{other}\ \right]$$

$$d\bar{w}(e_1, \dots, e_m) = \sum_i \frac{\partial}{\partial x^i} f_i$$

$$\int_M dw = \sum_{i=1}^{m} \int_{\{x^1\le 0\}} \frac{\partial}{\partial x^i} f^i(x^1, \dots x^m, 0, 0)\, dx^1 \cdots dx^m$$

$$= \int_{\{x^1\le 0\}} \frac{\partial}{\partial x^1} f^1(x^1, \dots x^m, 0, \dots 0)\, dx^1 \cdots dx^m + 0$$

$$\overset{\text{baby Stokes}}{=} \int_{\{x^1=0\}} f^1(x^1, \dots x^m, 0 \dots 0)\, dx^2 \cdots dx^m \quad \text{(baby Stokes)}$$



$$= \int_{\{x_1 = 0\}} \bar{w}(e_1, \dots, e_{m-1}) \qquad \left(\bar{w} = (\varphi^{-1})^* w \right)$$

$$= \int_{\partial M} w$$

$\boxed{\text{case 2}}$ $\quad w = \sum_\alpha \lambda_\alpha w^\alpha \qquad$ (w^α as in case 1 for some $(V^\alpha, \varphi^\alpha)$)

then $\quad \displaystyle\int_M dw = \sum_\alpha \int_M d(\lambda_\alpha w^\alpha) \overset{\text{by case 1}}{=} \sum_\alpha \int_{\partial M} \lambda_\alpha w^\alpha = \int_{\partial M} w$

∎