pf. Notation $f_i = \frac{\partial f}{\partial x^i}$, $f_{ij} = \frac{\partial f}{\partial x^i \partial x^j}$, ...

$$\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} = \frac{\partial}{\partial x^i}\langle f_j, f_l\rangle + \frac{\partial}{\partial x^j}\langle f_i, f_l\rangle - \frac{\partial}{\partial x^l}\langle f_i, f_j\rangle$$

$$= \underbrace{\langle f_i, f_{lj} - f_{jl}\rangle}_{0} + \underbrace{\langle f_j, f_{li} - f_{il}\rangle}_{0} + 2\langle \underbrace{f_l}_{\text{tangent}}, f_{ij}\rangle$$

$$= 2\langle f_l, f_{ij}^T\rangle \overset{\text{def } \Gamma_{ij}}{=} 2\left\langle f_l, \sum_k \Gamma_{ij}^k f_k\right\rangle$$

$$= 2\sum_k \Gamma_{ij}^k g_{lk}$$

$\times$ both side by g^{hl} and sum over l

$\left[\sum_l g^{hl} g_{lk} = \delta^h_k\right]$

Replace $h \to k$ ∎

Case $m=2$: Gauss' notation $E := g_{11}$, $F = g_{21}$, $G = g_{22}$

Christoffel symbols $\begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{12} & \Gamma^1_{22} \\ \Gamma^2_{11} & \Gamma^2_{12} & \Gamma^2_{22} \end{pmatrix} = \frac{1}{2D} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} E_1 & E_2 & 2F_2 - G_1 \\ 2F_1 - E_2 & G_1 & G_2 \end{pmatrix}$

Def'n 3.4 $M \subset \mathbb{R}^n$ m-dim. submanifold, $c: I \longrightarrow M$ curve,

$X: I \to \mathbb{R}^n$ C^1 tangent vector field on M along c (i.e $X(t) \in TM_{c(t)}$ for all $t \in I$)

Covariant derivative: $\frac{D}{dt} X : I \to \mathbb{R}^n$, $\frac{D}{dt} X(t) := \dot{X}(t)^T$ "tangential part"

Orthogonal projection $\mathbb{R}^n \to TM_{c(t)}$

X is parallel along c if $\frac{D}{dt} X(t) = 0$

for all $t \in I$, i.e $\dot{X}(t) \in TM_{c(t)}^{\perp}$

<u>Thm 3.5</u> M, c as above, $X, Y : I \longrightarrow \mathbb{R}^n$ C^1 tangent vector field along c.
$\lambda : I \longrightarrow \mathbb{R}$

Then:

(1) $\frac{D}{dt}(X+Y) = \frac{D}{dt}X + \frac{D}{dt}Y$

(2) $\frac{D}{dt}(\lambda X) = (\frac{d}{dt}\lambda)X + \lambda\frac{D}{dt}X$

(3) $\frac{d}{dt}g(X,Y) = g(\frac{D}{dt}X, Y) + g(X, \frac{D}{dt}Y)$

(4) If $f : U \to f(U) \subset M$ is a local param.

$\gamma : I \to U$ C^1 s.t. $c = f \circ \gamma$

$\xi : I \to \mathbb{R}^m$ C^1 $X(t) = df_{\gamma(t)}(\xi(t)) = \sum_{i=1}^{m} \xi^i(t) \frac{\partial}{\partial x^i}(\gamma(t))$

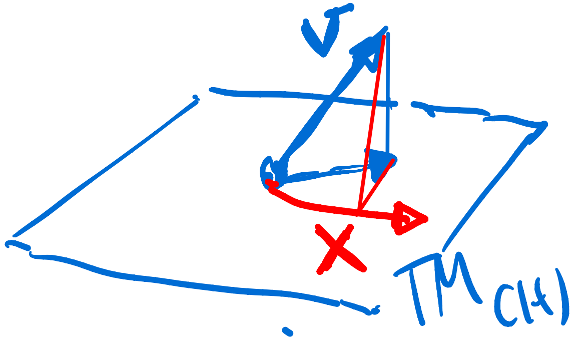
then $\frac{D}{dt}X = \sum_{i=1}^{m}\left[\dot{\xi}^k + \sum_{i,j=1}^{m} \xi^i \dot{\gamma}^j (\Gamma_{ij}^k \circ \gamma)\right] \frac{\partial f}{\partial x^k} \circ \gamma$

__proof__ (1) and (2) straightforward from def'n

when $\varphi : \mathbb{R} \to \mathbb{R}^n$ we denote $\dot{\varphi}$ or φ' ($d\varphi$)

(3) $$g(X,Y)' = g(\dot{X}, Y) + g(X, \dot{Y}) = g(\dot{X}^T, Y) + g(X, \dot{Y}^T)$$

X,Y are tangent



so $$\begin{cases} \langle v, X\rangle = \langle v^T, X\rangle \\ \langle v, Y\rangle = \langle v^T, Y\rangle \end{cases} \quad \forall v \in \mathbb{R}^n$$

(4) Denote $f_i := \frac{\partial f}{\partial x^i}$, $f_{ij} := \frac{\partial^2 f}{\partial x^j \partial x^i}$, $X = \sum_{i=1}^{m} \xi^i (f_i \circ \gamma)$

$$\left(\left(\xi^i (f_i \circ \gamma)\right)'\right)^T = \left(\dot{\xi}^i (f_i \circ \gamma)\right)^T + \left(\xi^i \sum_{j=1}^{m} (f_{ij} \circ \gamma)\dot{\gamma}^j\right)^T$$ take tangential parts

__Def'n of Γ^k_{ij}__ $$= \dot{\xi}^i (f_i \circ \gamma) + \xi^i \sum_{j,k=1}^{m} \dot{\gamma}^j (\Gamma^k_{ij} f_k) \circ \gamma$$

$$(f_{ij}(x))^T =: \sum_{k=1}^{m} \Gamma^k_{ij}(x) f_k(x)$$

Summing over i, we conclude (4) ∎

Remarks 1. (4) shows that $\frac{D}{dt}X$ is intrinsec

2. If X, Y are parallel along c, then $g_{c(t)}(X(t), Y(t))$ is constant in t $\quad \frac{d}{dt} g(X,Y) \underset{(3)}{=} g(\underbrace{\tfrac{D}{dt}X}_{=0}, Y) + g(X, \underbrace{\tfrac{D}{dt}Y}_{=0}) = 0$

We call X a C^k tangent vector field on $M \subset \mathbb{R}^n$ if $X : M \to \mathbb{R}^n$

$X = (X^1, \dots, X^n)$ X^i are C^k functions from M to $\mathbb{R}^n$

and $\boxed{X(p) \in TM_p}$ $\forall p \in M$ this is why they are called tangent!

X, Y tangent vf on M then we define the covariant derivative of Y with respect to X at $p \in M$

$$(D_Y X)(p) := \frac{D}{dt}(X \circ c)(0) \text{ for any } c : (-\varepsilon, \varepsilon) \to M \text{ s.t. } c(0) = p,\ c'(0) = Y(p)$$

This def'n is independent of choice of c

$X : M \to \mathbb{R}^n$ by chain rule (on submanifold)

differential $TM_p \to \mathbb{R}^n$

$$\left((X \circ c)'(0)\right)^T = \left((dX)_p\, \dot{c}(0)\right)^T = \left(dX_p \cdot Y(p)\right)^T \qquad (*)$$

Also if $\bar{X} : V \to \mathbb{R}^n$, $V \supset M$ open extends X (i.e. $\bar{X}|_M = X$)

$V \subseteq \mathbb{R}^n$

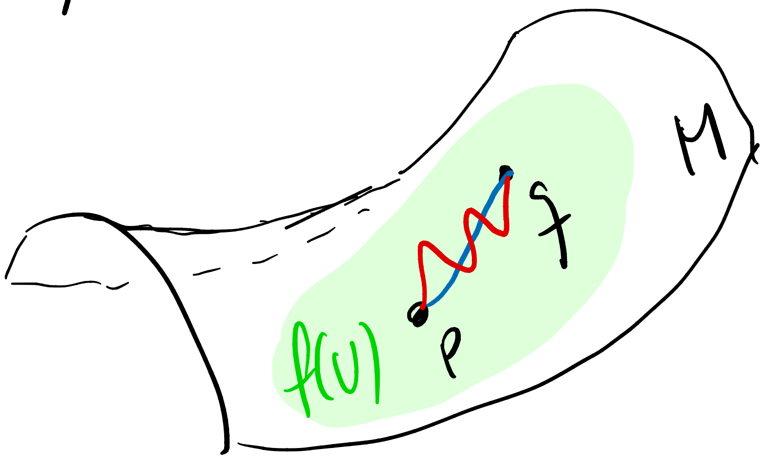
then similarly

$$D_Y X = d\bar{X}_p \cdot Y(p)$$

Analysis 2 differential $\mathbb{R}^n \to \mathbb{R}^n$

(*) shows this does not depend on the extension $\bar{X}$ of X employed

In order to motivate covariant derivative we now give the proof of Thm 3.10 (see the statement in the notes) but in a motivated way.



What is the shortest curve joining p and q?

To answer this: Calculus of Variations

Suppose that the shortest curve exists and is smooth (C^2)
(one proves this later!) $c : [a,b] \longrightarrow M$ $c(a) = p$ $c(b) = q$

Assume wlog. $|c'| = 1$ (reparametrize by arc length)

Suppose for simplicity p, q belong to the $f(U)$ f local param.

Let $\gamma : [a,b] \to U \subset \mathbb{R}^m$ st $c = f \circ \gamma$

given any $h : [a,b) \to \mathbb{R}^m$ of class C^2 such that $h(a) = h(b) = 0$

Consider $\bar{c}(\varepsilon, t) := f(\gamma + \varepsilon h)$ $\varepsilon \in \mathbb{R}$ ($|\varepsilon|$ small so that $\gamma + \varepsilon h \in U$)

Note that $\bar{c}(\varepsilon, a) = p$ and $\bar{c}(\varepsilon, b) = q$ $\forall \varepsilon$ as above

so $\bar{c}(\varepsilon, t)$ is a "competitor" curve.

$c(t)$ has minimal lenght among curves joining p, q

$\Rightarrow$ $L(\bar{c}(\varepsilon, \cdot)) \geq L(c)$ $\forall \varepsilon$ (with $|\varepsilon|$ small)

$$\Rightarrow 0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} L(\bar{c}(\varepsilon, \cdot)) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_a^b \sqrt{\langle \bar{c}_t(\varepsilon,t), \bar{c}_t(\varepsilon,t)\rangle}\, dt$$

$$c_t = \frac{\partial}{\partial t} c$$

$$= \int_a^b \frac{2\langle \frac{\partial}{\partial \varepsilon} \bar{c}_t(0,t), \bar{c}_t(0,t)\rangle}{2\sqrt{\langle \bar{c}_t(0,t), \bar{c}_t(0,t)\rangle}}\, dt$$

$$= \int_a^b \left\langle \frac{\partial}{\partial \varepsilon} \bar{c}_t(0,t),\ c_t'(t) \right\rangle dt \qquad |c_t(0,t)| = |c'(t)| = 1$$

integration by parts

$$= -\int_a^b \left\langle \frac{\partial}{\partial \varepsilon} \bar{c}(0,t),\ c_{tt}(t) \right\rangle dt + \underbrace{\left[\frac{\partial}{\partial \varepsilon} \bar{c}(0,t),\ c_t(t) \right]_{t=a}^{t=b}}_{=0}$$

$$\int_a^b \varphi' \psi = -\int_a^b \varphi \psi' + [\varphi \psi]_a^b$$

(with respect to variable t!)

$$= -\int_a^b \left\langle \underbrace{\frac{\partial}{\partial \varepsilon} \bar{c}(0,t)}, c''(t) \right\rangle$$

as we vary g we can generate any tangent vector field (vanishing on p and q)

The integral will be $= 0$ $\forall g$

if and only if $c''(t)$ is tangent $\forall t \in [a,b]$

$$\Longleftrightarrow \quad \boxed{\frac{D}{dt} c' = 0}$$

This is the geodesic eq'n. $\frac{D}{dt}$ appears naturally

We now define <u>parallel transport</u>. We need the following

<u>Thm</u> 3.6. $M \subset \mathbb{R}^n$ submanifold, $c: I \to M$ C^1 curve, $0 \in I$, $X_0 \in TM_{c(0)}$ $\Rightarrow$ $\exists$ unique parallel C^1 vector field X along c with $X(0) = X_0$

<u>Proof</u> We may assume I compact and $c(I) \subset f(U)$ for a local param. $f: U \to f(U) \subset M$. Write $c = f \circ \gamma$ $X = \sum_{i=1}^m \xi^i \frac{\partial f}{\partial x^i} \circ \gamma$ $(*)$

by Thm 3.5 (4) ξ is parallel iff.

$$\dot{\xi}^k + \sum_{i,j=1}^m \xi^i \dot{\gamma}^j (\Gamma^k_{ij} \circ \gamma) = 0 \qquad (k = 1, \dots, m)$$

This is a (linear) ODE with cont. coeff for ξ

$\Rightarrow$ $\exists !$ sol'n with $X(0) = X_0$

(We then use it to define X via $(*)$)

X is called <u>parallel transport</u> of X_0 along c

Rmk 2 given before shows that for a curve c from p to q in M the parallel transport along c defines an <u>isometry</u>

$$(TM_p, g_p) \longrightarrow (TM_q, g_p)$$

This isometry depends typically on the curve c joining p and q

---

Reparam.

$$\begin{tikzcd} & M & \\ I \arrow[ur, "c"] \arrow[rr, "\varphi"] & & \tilde{I} \arrow[ul, "\tilde{c}"'] \end{tikzcd}$$

$$X(t) = \tilde{X}(\varphi(t)) = \tilde{X}(s)$$

$$\left(\frac{d}{dt} X\right)^T = \left(\frac{d}{ds} \tilde{X}(\varphi(t))\right)^T \dot{\varphi}(t)$$

$$\frac{D}{dt} X(t) = \frac{D}{ds} \tilde{X}(\varphi(t)) \underbrace{\dot{\varphi}(t)}_{\neq 0}$$

So, X parallel along c $\iff$ $\tilde{X}$ is parallel along $\tilde{c}$