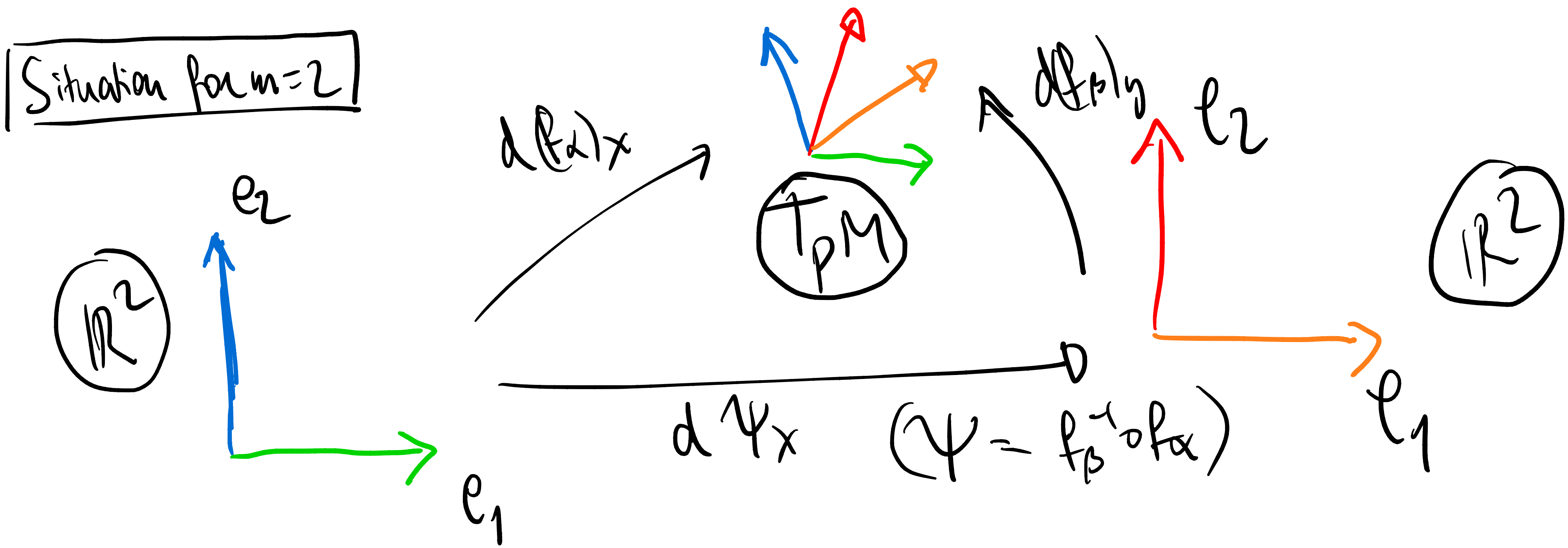
Prop 2.10 An m-dim submanifold $M \subset \mathbb{R}^{m+1}$ is orientable iff $\exists$ cont. unit normal vector field $N : M \to S^m$ (i.e. $N(p) \in T_pM^{\perp}$ $\forall p \in M$)

Pf. ($\Rightarrow$) M orientable $\Leftrightarrow$ $\exists \{f_\alpha : U_\alpha \to M\}_{\alpha \in A}$ s.t. $\bigcup_{\alpha \in A} f_\alpha(U_\alpha) = M$, with $\det(df_\beta^{-1} \circ f_\alpha) > 0$. Define $\nu_\alpha : U_\alpha \to S^m$ along f_α satisfying

$$\nu_\alpha(x) \in T M_{f_\alpha(x)}^{\perp} \quad \& \quad \det\left(\frac{\partial f_\alpha}{\partial x^1}, \ldots, \frac{\partial f_\alpha}{\partial x^m}, \nu_\alpha\right) > 0$$

we must show $\quad f_\alpha(x) = f_\beta(y) = p \Rightarrow \nu_\alpha(x) = \nu_\beta(y)$



Now, $(f_\alpha)_x$ and $(f_\beta)_y$ give the same N $\iff$

$\det(a_{ij}) > 0$ where $\frac{\partial f_\beta}{\partial y_j}(y) = a_{ij} \frac{\partial f_\alpha}{\partial x_j}(x)$ $\iff$ $d(f_\beta)_y e_i = a_{ij} d(f_\alpha)_x e_j$

$$e_i = a_{ij} (df_\beta)_y^{-1} (df_\alpha)_x$$
$$= a_{ij}\, d\Psi_x\, e_j$$

$\boxed{d\Psi_x = d(f_\beta)_y^{-1} (df_\alpha)_x \ \& \ (df_\beta)_y \text{ is isomorphism } \mathbb{R}^m \to T_pM} \implies$

Hence, $a_{ij} = d\Psi_x^{-1}$ and $\det(a_{ij}) > 0$ (since by assumption $\det(d\Psi_x) > 0$)

$\Leftarrow$) Conversely if N is given, given a local param.

$\{ f_\alpha : U_\alpha \to M \}_{\alpha \in A}$ s.t $\bigcup_{\alpha \in A} f_\alpha(U_\alpha) = M$

define $\tilde{f}_\alpha := \begin{cases} f_\alpha & \text{if } \det\left(\frac{\partial f_\alpha}{\partial x^1}, \dots, \frac{\partial f_\alpha}{\partial x^m}, N\right) > 0 \\ f_\alpha(-x_1, x_2, \dots, x_m) & \text{if } \quad '' \quad < 0 \end{cases}$

$\blacksquare$

Thm 2.11 (Jordan-Brouwer separation theorem)

$\emptyset \neq M \subset \mathbb{R}^{m+1}$ m-dim compact connected submanifold

$\Rightarrow$ $\mathbb{R}^{m+1} \setminus M$ has exactly 2 connected components A, B

with $M = \partial A = \partial B$, and (hence) M is orientable.

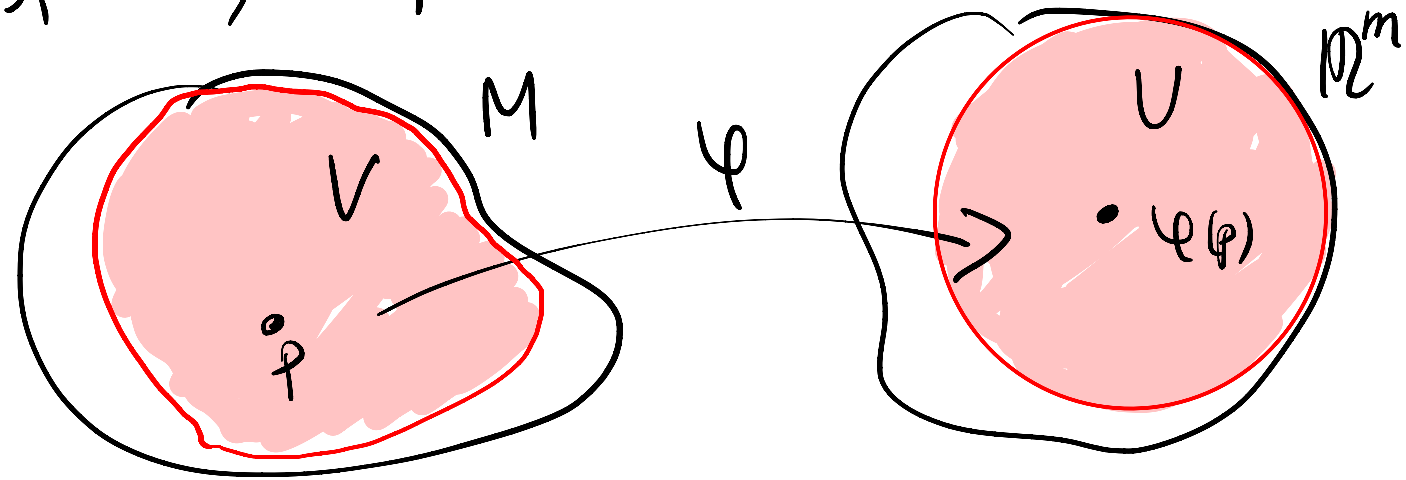
<u>preliminary</u> Given $p \in M$ $\exists V_p$ open neighborhood and

φ_p submanifold chart such that

$$\varphi_p(M \cap V_p) = B_1(0) \cap \{x_n = 0\}$$

$$\varphi_p(V_p) = B_1 \quad , \quad$$

$(n := m+1)$ Indeed,



let $\varphi : \underset{\underset{M}{\cap}}{V} \longrightarrow \underset{\underset{\mathbb{R}^n}{\cap}}{U}$

submanifold chart near p;

define $\bar{s} := \sup \{ s>0 \mid B_s(\varphi(p)) \subset U$

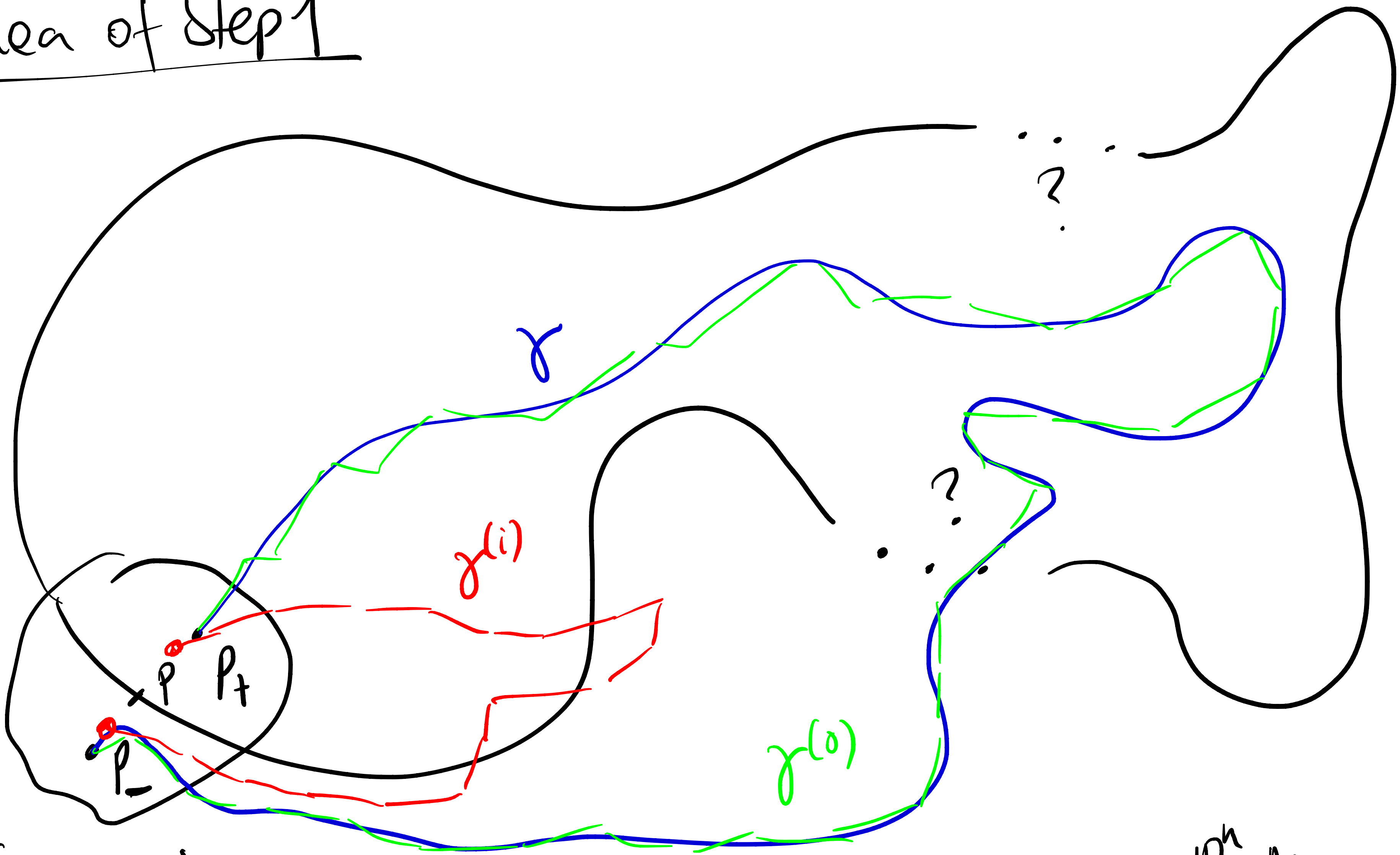
and take $\varphi_p := \left. \dfrac{\varphi - \varphi(p)}{\bar{s}} \right|_{\varphi^{-1}(B_{\bar{s}}(\varphi(p)))}$

Proof of thm 2.11 Step 1 Take $p \in M \Rightarrow$

$V_p \setminus M$ is diffeomorfic to $B_1 \setminus \{x_n = 0\}$ (φ_p is the diffeomorphism) and hence it has 2 connected components. Let us show that they belong to different connected components of $\mathbb{R}^n \setminus M$

$(n = m+1)$

Idea of Step 1



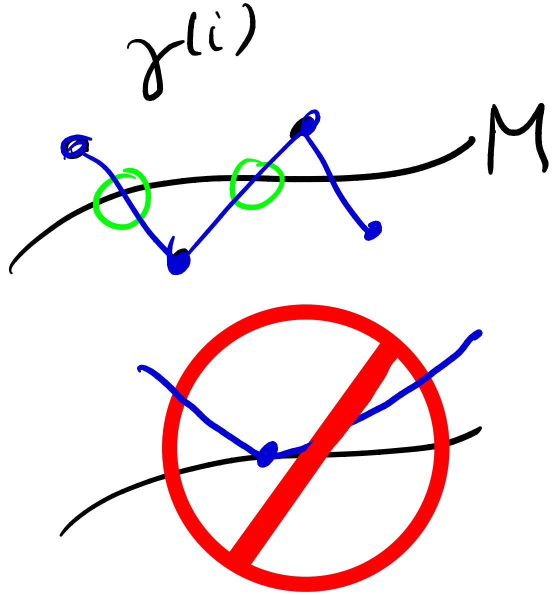
let $p_\pm^\varepsilon := \varphi_p^{-1}(\pm \varepsilon e_n)$

($\varepsilon > 0$ to be chosen)

Suppose by contradiction $\exists \gamma : [a,b] \to \mathbb{R}^n \setminus M$ such that $\gamma(a) = p_+^\varepsilon$, $\gamma(b) = p_-^\varepsilon$

We will construct a sequence piecewise smooth curves $\gamma^{(i)}$, $0 \le i \le N$, s.t.

- $\gamma(i)$ joins p_+^{ε} and p_-^{ε} for all $i = 0, \dots, N$
- $\gamma^{(0)}$ never intersects M
- $\gamma^{(N)} \subset V_p$
- $\gamma^{(i)}$ always <u>intersects transversely</u> M
- $\#(\gamma^{(i)} \cap M) \bmod 2$ is the same for all $i = 0, \dots, N$



This leads to a contradiction since then $\varphi_p \circ \gamma^{(N)}$ would be a curve $[a,b] \longrightarrow B_1 \setminus \{x_n = 0\}$ joining εe_n and $-\varepsilon e_n$ and intersecting an even number of times $x_n = 0$!

Construction of $\gamma^{(i)}$: for all $q \in M$ let

$$r_q := \sup\{ r > 0 \mid B_r(q) \subset V_q \}$$

The family of open balls $\left\{ B_{\frac{r_q}{100}}(q) \mid q \in M \right\}$ covers q, and — since M is compact — $\exists$ finite subcover

$$M \subset \bigcup_{\alpha \in \text{finite set}} B_{\frac{r_{q_\alpha}}{100}}(q_\alpha)$$

Let $$\delta := \min\left(\frac{\operatorname{dist}(\gamma([a,b]) \setminus V_p, M)}{100}, \min_\alpha \frac{r_{q_\alpha}}{100} \right)$$

Fix $a = t_1 < t_2 < \cdots < t_K = b$ such that

$|\gamma(t_{j+1}) - \gamma(t_j)| < \delta$ Let $D := \operatorname{diam}(\gamma([a,b]))$

Suppose (after traslation) that $p = 0$ and define, for $i \geq 0$, $j = 1, \ldots K$,

$$\mathbb{R}^n \setminus M \ni \gamma_j^{(i)} := \begin{cases} p_+^\varepsilon = \gamma(t_1) & j = 1 \\ (1 - \delta/D)^i \gamma(t_j) + \delta y_j^{(i)}, & j = 2, \ldots, K \\ p_-^\varepsilon = \gamma(t_M) & j = K \end{cases}$$

where $y_j^{(i)}$ is <u>any</u> point in B_1 chosen so that $\gamma_j^{(i)} \notin M$.

<u>Choose</u> $\varepsilon > 0$ so that $|p_\pm^\varepsilon| < \delta$

By construction
$$\begin{cases} \| \gamma_j^{(i+1)} - \gamma_j^{(i)} \| \leq 3\delta \\ \| \gamma_{j+1}^{(i)} - \gamma_j^{(i)} \| \leq 3\delta \end{cases} \quad \forall i, j$$

Now, for all $i \geq 0$, given the (ordered by j!) sequence of pts $\gamma_1^{(i)}, \gamma_2^{(i)}, \dots, \gamma_K^{(i)}$ and $1 \leq \ell \leq K$ we define

$$\gamma_j^{(i,\ell)} = \begin{cases} \gamma_j^{(i+1)} & \text{if } j \leq \ell \\ \gamma_j^{(i)} & \text{if } j > \ell \end{cases}$$

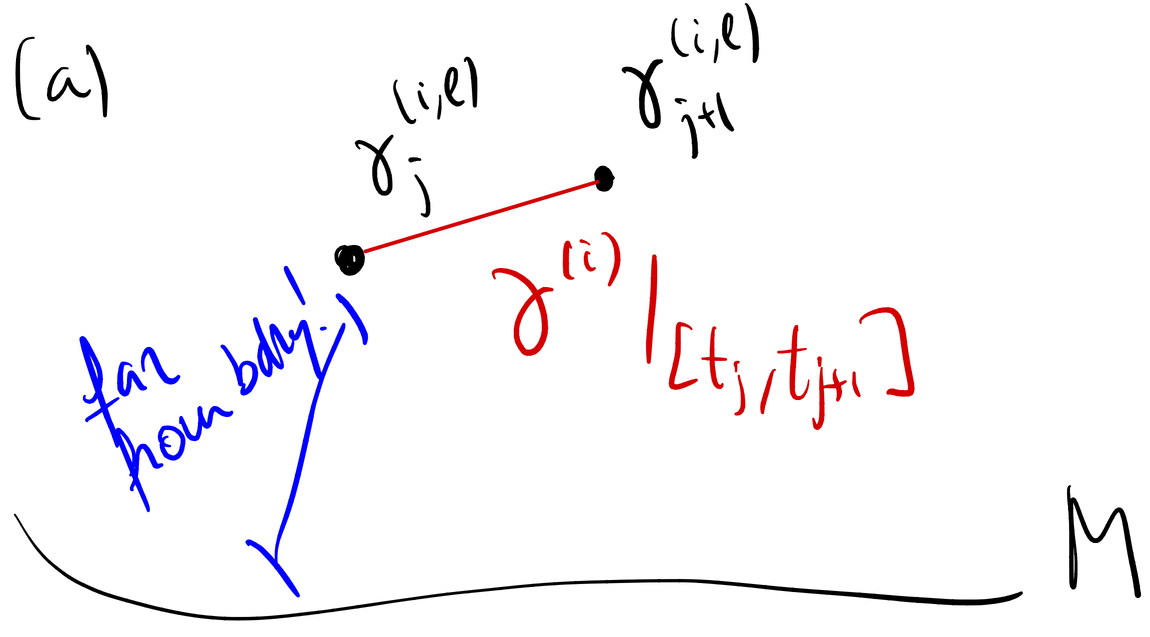
Now for each (i,ℓ) define the curve $\gamma^{(i,\ell)} : [a,b] \to \mathbb{R}^n$ as follows

(a) If $\operatorname{dist}(\gamma_j^{(i,\ell)}, M) > 20\delta$ $\quad \gamma_j^{(i)}\Big|_{[t_j, t_{j+1}]}$ is the segment joining $\gamma_j^{(i,\ell)}$ and $\gamma_{j+1}^{(i,\ell)}$

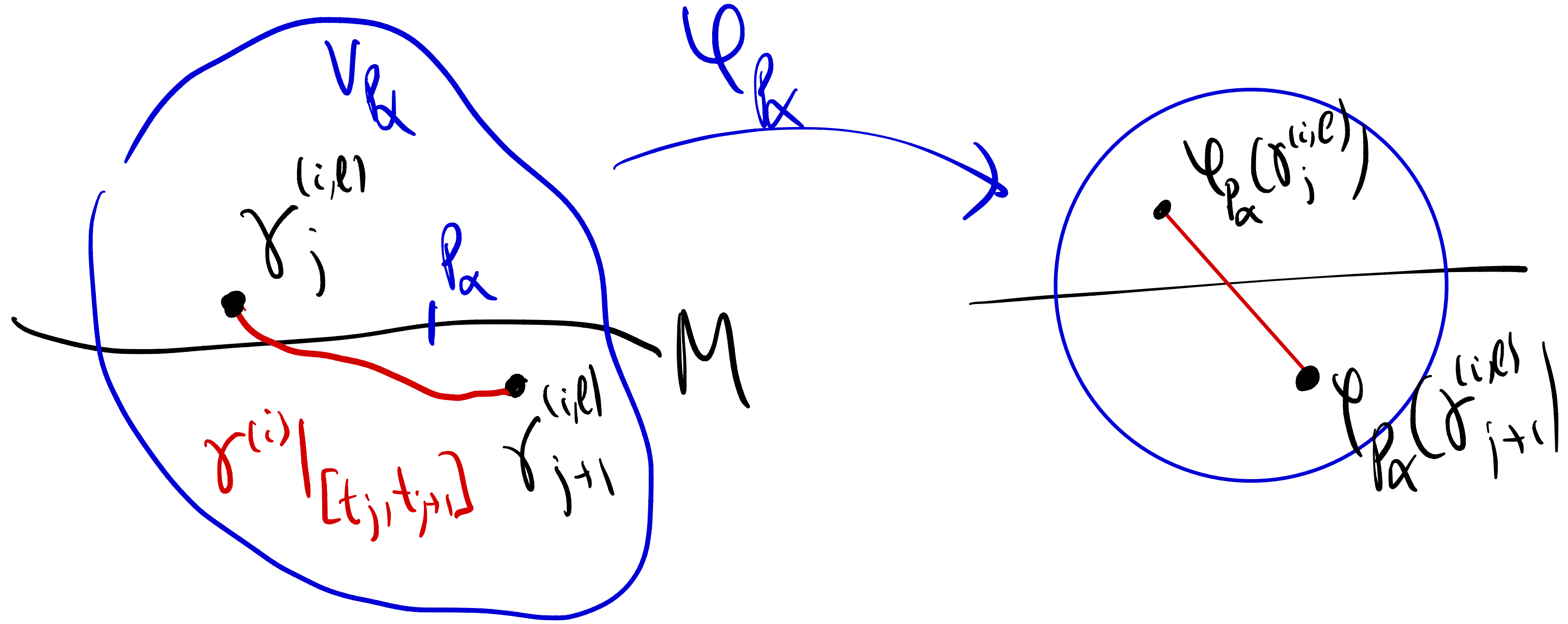
(b) If $\operatorname{dist}(\gamma_j^{(i,\ell)}, M) \leq 20\delta$ then choose α such that $\gamma_j^{(i,\ell)}, \gamma_{j+1}^{(i,\ell)} \in V_{p_\alpha}$ and $\gamma_j^{(i)}\Big|_{[t_j, t_{j+1}]} := \varphi_{p_\alpha}^{-1} \circ \left[\begin{array}{c} \text{segment joining} \\ \varphi_{p_\alpha}(\gamma_j^{(i)}) \text{ \& } \varphi_{p_\alpha}(\gamma_{j+1}^{(i)}) \end{array} \right]$

Picture

(a)


$\gamma_j^{(i,\ell)}$
$\gamma_{j+1}^{(i,\ell)}$
$\gamma^{(i)}|_{[t_j, t_{j+1}]}$
far from bdry!
M

(b)


V_{p_α}
φ_{p_α}
$\gamma_j^{(i,\ell)}$
p_α
M
$\gamma^{(i)}|_{[t_j, t_{j+1}]}$
$\gamma_{j+1}^{(i,\ell)}$
$\varphi_{p_\alpha}(\gamma_j^{(i,\ell)})$
$\varphi_{p_\alpha}(\gamma_{j+1}^{(i,\ell)})$

Now let us show that

$$\#\left(\gamma^{(i,\ell)}([a,b]) \cap M\right) = \#\left(\gamma^{(i',\ell')}([a,b]) \cap M\right) \mod 2$$

for all (i,ℓ), (i',ℓ')

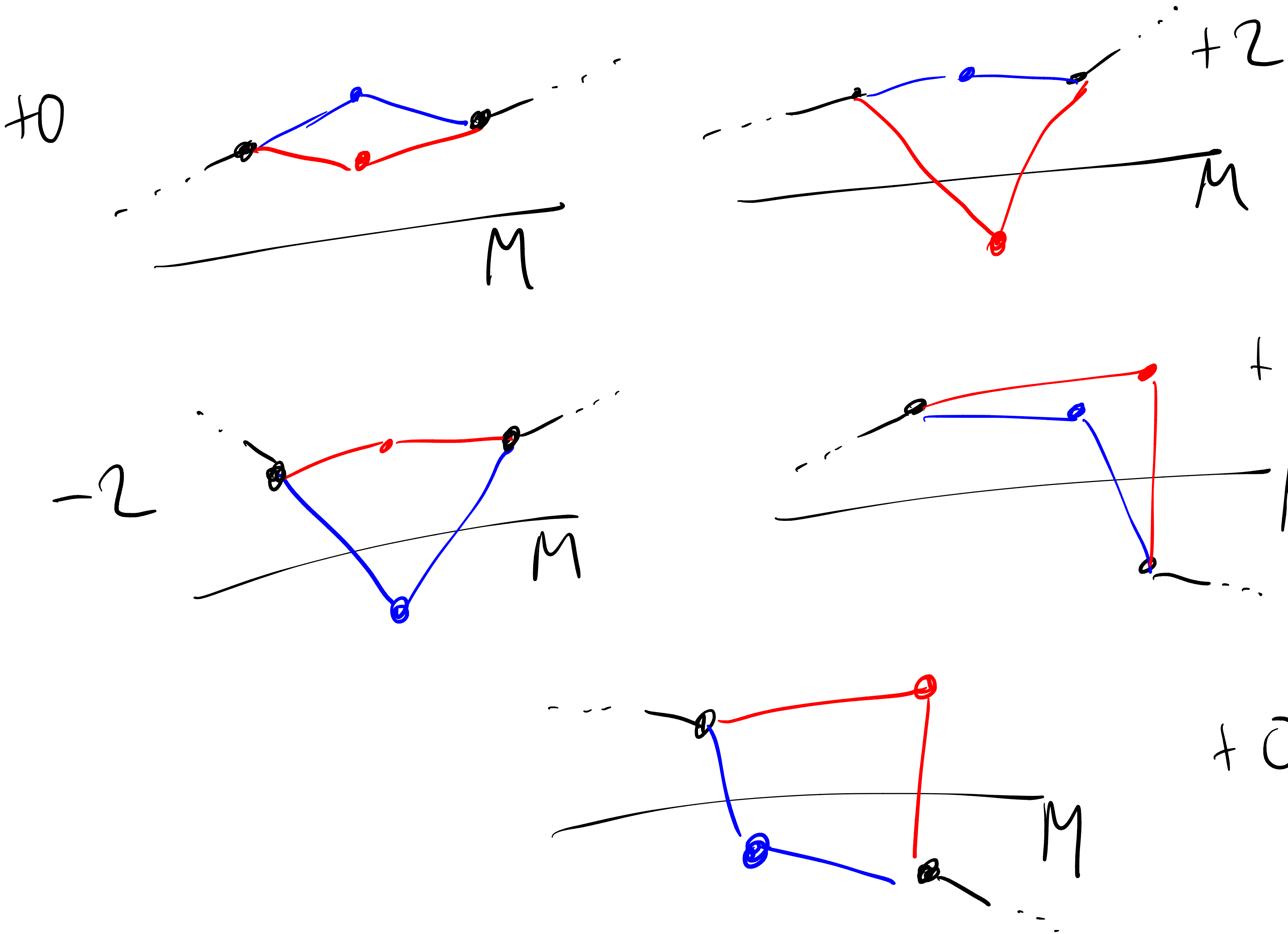
It is enough to show it for $(i,\ell) \rightsquigarrow (i,\ell+1)$

and $(i,k) \rightsquigarrow (\ell+1, 1)$

In such situation only 1 point changes between $\gamma^{(i,\ell)}$ and $\gamma^{(i',\ell')}$, $\gamma^{(i)}_{\ell}$ changes to $\gamma^{(i+1)}_{\ell}$

Hence the difference in the intersection number ~~is either~~ is always even (use φ_{x_p} when close to bdry!)

OLD point NEW point



This completes Step 1!

Step 2 Let us show that $\mathbb{R}^n \setminus M$ has exactly 2 connected components (Step 1 shows at least 2)

Indeed, given any two $p, q \in M$, by step 1,

$p \in \partial A_p \cap \partial B_p$ & $q \in \partial A_q \cap \partial B_q$ for

connected components $A_p \neq B_p$ and $A_q \neq B_q$ of $\mathbb{R}^n \setminus M$

Now, since M is connected $\exists$ curve $c : [a,b] \to M$ from p to q

Let $N : [a,b] \to S^m$ be a continuous unit normal to M along the curve c. Now for $\varepsilon > 0$ small enough the curves

$$t \longmapsto c(t) + \varepsilon N(t) \quad \text{are in} \quad \mathbb{R}^n \setminus M$$

$\Rightarrow$ either $A_p \equiv A_q$ and $B_p = B_q$

or $A_p \equiv B_q$ and $A_q = B_p$

$\Rightarrow$ $\mathbb{R}^n \setminus M$ has 2 connected components and since M is bdd only one of them can be unbounded. We call the bdd component interior and the unbdd one exterior

Now we define the exterior normal $N : M \to S^n$ as the normal such that $x + \varepsilon N(x)$ belong to the exterior. (resp. interior normal). Then N is continuous on M and hence (by Prop. 2.10) M is orientable ∎