PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 1

Exercise 1. Let X and Y be independent variables such that X and Y are Poisson distributed with parameters $\lambda > 0$ and $\mu > 0$ respectively, i.e. $\mathbb{P}(X = k) = e^{-\lambda}\,\lambda^k/k!$ and $\mathbb{P}(Y = l) = e^{-\mu}\,\mu^l/l!$ for $k, l \in \mathbb{N}_0$. Show that $X + Y$ is Poisson distributed with parameter $\lambda + \mu$, i.e.

$$\mathbb{P}(X + Y = n) = e^{-(\lambda+\mu)}\,\frac{(\lambda + \mu)^n}{n!}$$

for $n \in \mathbb{N}_0$ and determine $\mathbb{P}(X = k \mid X + Y = n)$ for all $k, n \in \mathbb{N}_0$.

Exercise 2. Let U and V be independent random variables which are geometrically distributed with respective parameters $p \in (0, 1)$ and $q \in (0, 1)$, i.e. we have $\mathbb{P}(U = k) = (1 - p)^{k-1}p$ and $\mathbb{P}(V = l) = (1 - q)^{l-1}q$ for $k, l \in \mathbb{N}$. Show that $U \wedge V$ is geometrically distributed with parameter $r := 1 - (1 - p)(1 - q)$, i.e.

$$\mathbb{P}(U \wedge V = n) = (1 - r)^{n-1}r$$

for all $n \in \mathbb{N}$.

Exercise 3. Let X and Y be independent and let both be uniformly distributed on $\{\pm 1\}$. Also let $Z = XY$. Show that X, Y, Z are pairwise independent but not independent.

Exercise 4. Let $E_1, \dots, E_n$ be countable sets and let $P_i \colon \mathcal{P}(E_i) \to [0, 1]$ be a probability measure on the power set of E_i for all $i = 1, \dots, n$. Let $E = E_1 \times \cdots \times E_n$ and define $P \colon \mathcal{P}(E) \to [0, 1]$ by

$$P(A) = \sum_{(x_1,\dots,x_n)\in A} P_1(\{x_1\}) \cdots P_n(\{x_n\}) \quad \text{for } A \subset E\,.$$

Show that P is a probability measure on E. Also for $i = 1, \dots, n$ define the random variable $X_i(x_1, \dots, x_n) = x_i$. Show that $X_1, \dots, X_n$ are independent.

Exercise 5. Let (A_n) and (B_n) be sequences of events on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(i) Suppose that $A_n \subset A_{n+1}$ for all $n \geq 1$. Show that $\mathbb{P}(A_n) \to \mathbb{P}(\cup_m A_m)$ as $n \to \infty$.

(ii) Suppose now that $A_n \supset A_{n+1}$ for all $n \geq 1$. In this case, show that $\mathbb{P}(A_n) \to \mathbb{P}(\cap_m A_m)$ as $n \to \infty$.

(iii) We now consider a general sequence (B_n). Recall the definitions $\liminf_{n\to\infty} B_n = \cup_{n\geq 1} \cap_{m\geq n} B_m$ and $\limsup_{n\to\infty} B_n = \cap_{n\geq 1} \cup_{m\geq n} B_m$. Show that

$$\mathbb{P}\left(\liminf_{n\to\infty} B_n\right) \leq \liminf_{n\to\infty} \mathbb{P}(B_n) \leq \limsup_{n\to\infty} \mathbb{P}(B_n) \leq \mathbb{P}\left(\limsup_{n\to\infty} B_n\right)\,.$$

Exercise 6. This question is about π-systems and Dynkin systems.

(i) Show that $\mathcal{A} = \{[0, a] \colon a \in [0, 1]\}$ is a π-system generating $\mathcal{B}([0, 1])$.

(ii) Prove that $\mathcal{A}' = \{(-\infty, a_1] \times \cdots \times (-\infty, a_d] \colon a_1, \ldots, a_d \in \mathbb{R}\} \cup \{\mathbb{R}^d\}$ is a π-system generating the σ-algebra $\mathcal{B}(\mathbb{R}^d)$.

(iii) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and fix $A \in \mathcal{F}$. Show that

$$\mathcal{D} := \{B \in \mathcal{F} \colon \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)\}$$

is a Dynkin system.

(iv) Give an example of a Dynkin system which is not a σ-algebra.

Submission of solutions. Hand in by 27/09/2021 5 p.m. (online) following the instructions on the course website

`https://metaphor.ethz.ch/x/2021/hs/401-3601-00L/`

The exercise classes are listed below; the Zoom meeting details are given on the course website shown above.

Time	**Room**	**Assistant**
Tuesday 2 p.m. – 3 p.m.	HG F 26.5	Matthis Lehmkuehler
Tuesday 2 p.m. – 3 p.m.	ML H 41.1	Luca Pelizzari
Tuesday 3 p.m. – 4 p.m.	Zoom	Daniel Contreras Salinas
Tuesday 3 p.m. – 4 p.m.	ML H 41.1	Genc Kqiku