

**PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 10**

Exercise 1. Let (M_n) be a non-negative submartingale with respect to a filtration (\mathcal{F}_n) and let us define $M_n^* = \max\{M_0, \dots, M_n\}$.

(i) Show that Doob's maximal inequality holds, i.e. that for all $x > 0$,

$$x \cdot \mathbb{P}(M_n^* \geq x) \leq \mathbb{E}(M_n; M_n^* \geq x) \leq \mathbb{E}(M_n).$$

(ii) Deduce Doob's L^p inequality, i.e. show that for all $p > 1$ we have

$$\mathbb{E}((M_n^*)^p) \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}(M_n^p).$$

(iii) Let (X_n) be a sequence of independent centered random variables and let $S_n = X_1 + \dots + X_n$ and $S_0 = 0$. Use part (i) to show that

$$\mathbb{P}\left(\max_{1 \leq i \leq n} |S_i| \geq \lambda\right) \leq \frac{\mathbb{E}(S_n^2)}{\lambda^2}.$$

Exercise 2. Let (X_n) be a sequence of random variables and define a filtration (\mathcal{F}_n) by $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$. Also let $\mathcal{F}_\infty = \sigma(\cup_{n \geq 0} \mathcal{F}_n)$. Let T be a stopping time with respect to (\mathcal{F}_n) . Recall that

$$\mathcal{F}_T = \{A \in \mathcal{F}_\infty : A \cap \{T = n\} \in \mathcal{F}_n \forall n \geq 0\}.$$

Show that $\mathcal{F}_T = \sigma(T, X_{T \wedge n} : n \geq 0)$.

Exercise 3. Let (M_n) be a martingale with respect to a filtration (\mathcal{F}_n) and let S and T be a stopping times. Show that

$$\mathbb{E}(M_{n \wedge S} | \mathcal{F}_T) = M_{n \wedge S \wedge T} \quad \text{for all } n \geq 0 \text{ a.s.}$$

Exercise 4. Let (X_n) be a sequence of random variables.

- (i) Let us suppose that (X_n) converges in probability to a random variable X . Show that (X_n) converges to X in distribution, i.e. the law of X_n converges weakly to the law of X .
- (ii) Suppose that $c \in \mathbb{R}$ and $X = c$ almost surely. Show that if (X_n) converges in distribution to a X then (X_n) converges in probability to c .
- (iii) Give an example of random variables (X_n) and X such that X_n converges in distribution but not in probability to X as $n \rightarrow \infty$.

Submission of solutions. Hand in by 29/11/2021 5 p.m. (online) following the instructions on the course website

<https://metaphor.ethz.ch/x/2021/hs/401-3601-00L/>

The exercise classes are listed below; the Zoom meeting details are given on the course website shown above.

Time	Room	Assistant
Tuesday 2 p.m. – 3 p.m.	HG F 26.5	Matthis Lehmkuehler
Tuesday 2 p.m. – 3 p.m.	ML H 41.1	Luca Pelizzari
Tuesday 3 p.m. – 4 p.m.	Zoom	Daniel Contreras Salinas
Tuesday 3 p.m. – 4 p.m.	ML H 41.1	Genc Kqiku