PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 11

Exercise 1. Suppose that $B \sim B(n,p)$ is Binomially distributed, $P \sim P(\lambda)$ is Poisson distributed, $N \sim N(\mu, \sigma^2)$ follows a normal distribution and C has a Cauchy distribution, i.e. it has law $\mu_C(dx) = dx/(\pi(1+x^2))$ on $\mathbb{R}$.

(i) Compute the characteristic functions of B, P, N and C. Hint: For the computation of C you might want to use the residue theorem from complex analysis.

(ii) Show that if $C_1, \dots, C_n$ are independent Cauchy distributed random variables then $(C_1 + \cdots + C_n)/n$ is again a Cauchy distributed random variable.

(iii) Fix $\lambda > 0$. For each $n \geq 1$ let X_n have a Binomial distribution with parameters $(n, \lambda/n)$ and let X have a Poisson distribution with parameter λ. Show that X_n converges in distribution to X.

Exercise 2. In this question, you may admit the fact that there is a unique probability measure μ_n on the unit sphere $\{(x_1, \dots, x_n) \in \mathbb{R}^n \colon x_1^2 + \cdots + x_n^2 = 1\}$ such that $R_* \mu_n = \mu_n$ for all $R \in O(n)$ (the orthogonal group). Suppose that $X^{(n)} = (X_1^{(n)}, \dots, X_n^{(n)}) \sim \mu_n$.

(i) Show that $X^{(n)}$ has the same law as

$$N^{(n)} = \left(\frac{N_1}{\sqrt{N_1^2 + \cdots + N_n^2}}, \dots, \frac{N_n}{\sqrt{N_1^2 + \cdots + N_n^2}} \right)$$

where (N_i) is a sequence of i.i.d. $N(0,1)$ random variables.

(ii) Use this to show that for each fixed $i \in \mathbb{N}$ we have that $\sqrt{n} X_i^n$ converges in distribution to $N \sim N(0,1)$ as $n \to \infty$.

Exercise 3. Let (X_n) be a sequence of random variables and suppose that $\phi \colon [0,\infty) \to [0,\infty)$ is measurable and satisfies $\phi(x) \to \infty$ as $x \to \infty$. Assume further that $\sup_n \mathbb{E}(\phi(|X_n|)) < \infty$. Show that the sequence (X_n) is tight.

Exercise 4. Let X be a symmetric stable distribution with exponent $\alpha \in (0,2)$, i.e. $\mathbb{E}(e^{i\theta X}) = e^{-|\theta|^\alpha}$ for all $\theta \in \mathbb{R}$. Let (X_i) be a sequence of i.i.d. random variables with

$$\mathbb{E}(e^{i\theta X_1}) = 1 - |\theta|^\alpha + o(|\theta|^\alpha)$$

as $\theta \to 0$. Show that $(X_1 + \cdots + X_n)/n^{1/\alpha}$ converges in distribution to X as $n \to \infty$.

Exercise 5*. Let (X_n) and X be random variables and suppose that $\mathbb{E}(X_n^p) \to \mathbb{E}(X^p)$ as $n \to \infty$ for all $p \in \mathbb{N}$. Also suppose that $\mathbb{E}(e^{\epsilon|X|}) < \infty$ for some $\epsilon > 0$.

(i) Show that (X_n) is tight
(ii) Consider any subsequence (n_k). Show that there is a further subsequence (n_{k_l}) and a probability measure μ such that the law of $X_{n_{k_l}}$ converges weakly to μ as $l \to \infty$.
(iii) Suppose that $Y \sim \mu$. Show that $\mathbb{E}(X^p) = \mathbb{E}(Y^p)$ for all $p \geq 0$.
(iv) Deduce that $X =_d Y$. Hint: Show that the characteristic functions of X and Y extend to analytic functions on the domain $D = \{z \in \mathbb{C} \colon |\Im(z)| < \epsilon\}$ and argue that they are equal.
(v) Deduce that X_n converges in distribution to X as $n \to \infty$

Submission of solutions. Hand in by 06/12/2021 5 p.m. (online) following the instructions on the course website

`https://metaphor.ethz.ch/x/2021/hs/401-3601-00L/`

The exercise classes are listed below; the Zoom meeting details are given on the course website shown above.

Time	Room	Assistant
Tuesday 2 p.m. – 3 p.m.	HG F 26.5	Matthis Lehmkuehler
Tuesday 2 p.m. – 3 p.m.	ML H 41.1	Luca Pelizzari
Tuesday 3 p.m. – 4 p.m.	Zoom	Daniel Contreras Salinas
Tuesday 3 p.m. – 4 p.m.	ML H 41.1	Genc Kqiku