PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 12

Exercise 1. Let (X_i) be i.i.d. random variables with symmetric stable distribution of parameter $\alpha \in (0,2)$.

(i) Find the distribution of $n^{-1/\alpha}(X_1 + \cdots + X_n)$.

(ii) Does $n^{-1/2}(X_1 + \cdots + X_n)$ converge in distribution (to some limit) as $n \to \infty$?

Exercise 2. Let $(X^{(n)})$ and X be random variables with values in $\mathbb{R}^d$. Show $X^{(n)} \to X$ in distribution as $n \to \infty$ if and only if for all $\theta \in \mathbb{R}^d$ we have $\theta \cdot X^{(n)} \to \theta \cdot X$ in distribution as $n \to \infty$.

Exercise 3. Let $X, Y, (X_n)$ and (Y_n) be random variables defined on the same probability space such that (X_n) converges in distribution to X and Y_n converges in distribution to Y as $n \to \infty$.

(i) Show that it is not necessarily true that $X_n + Y_n$ converges in distribution to $X + Y$ as $n \to \infty$.

(ii) If $Y = c$ a.s. for some $c \in \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, then show that $g(X_n, Y_n)$ converges in distribution to $g(X, c)$ as $n \to \infty$.

Exercise 4. Let (X_n) be a sequence of i.i.d. centered random variables with $\mathbb{E}(X_1^2) \in (0, \infty)$. Show that the sequence given by

$$Y_n := \frac{\sum_{k=1}^n X_k}{1 + (\sum_{k=1}^n X_k^2)^{1/2}}$$

converges in distribution as $n \to \infty$ and identify its limit. Hint: Use Exercise 3.

Exercise 5*. Let us define sets $D_n \subset [0,1]$ for $n \geq 0$ iteratively as follows. Let $D_0 = [0,1]$ and $D_{n+1} = (D_n/3) \cup (2/3 + D_n/3)$; the set $\cap_{n \geq 0} D_n$ is called the Cantor set. Let

$$\mu_n = \frac{1}{\lambda(D_n)} \lambda|_{D_n}$$

where λ denotes the Lebesgue measure. Show that μ_n converges weakly to a limiting probability measure μ. Hint: Show that if (ϵ_i) are i.i.d. Bernoulli random variables with parameter $1/2$ and $U \sim U(0,1)$ is independent of them then for $n \geq 0$,

$$U/3^n + \sum_{i=1}^n 2\epsilon_i/3^i \sim \mu_n \, .$$

Submission of solutions. Hand in by 13/12/2021 5 p.m. (online) following the instructions on the course website

`https://metaphor.ethz.ch/x/2021/hs/401-3601-00L/`

The exercise classes are listed below; the Zoom meeting details are given on the course website shown above.

Time	**Room**	**Assistant**
Tuesday 2 p.m. – 3 p.m.	HG F 26.5	Matthis Lehmkuehler
Tuesday 2 p.m. – 3 p.m.	ML H 41.1	Luca Pelizzari
Tuesday 3 p.m. – 4 p.m.	Zoom	Daniel Contreras Salinas
Tuesday 3 p.m. – 4 p.m.	ML H 41.1	Genc Kqiku