PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 13

Exercise 1. Let (X_i) be i.i.d. random variables and let $S_n = X_1 + \cdots + X_n$. Consider the following list of settings.

(i) X_1 is Poisson distributed with parameter $\lambda > 0$.
(ii) $X_1 \sim B(1, p)$ for $p \in (0, 1)$.
(iii) X_1 is exponentially distributed with rate $\lambda > 0$.
(iv) $X_1 \sim N(0, \sigma^2)$ for $\sigma > 0$.

In each of the cases listed above determine the limit $\lim_{n\to\infty} n^{-1} \log \mathbb{P}(S_n > xn)$ for all $x \in \mathbb{R}$ whenever it exists.

Exercise 2. Let (X_n) be a simple symmetric random walk on $\mathbb{Z}$ started at 0. Also let $M_n = \max\{k \le n : X_{2k} = 0\}$. The goal of this question will be to show that

$$M_n/n \to \sin^2(\pi U/2) \quad \text{in distribution as } n \to \infty$$

where $U \sim U(0, 1)$. The law of $\sin^2(\pi U/2)$ is called the arcsine distribution.

(i) Show that
$$\mathbb{P}(X_{2k} = 0) = \frac{1 + o(1)}{\sqrt{\pi k}} \quad \text{as } k \to \infty .$$
(ii) Let $P_k(a, b) = \{x \in \mathbb{Z}^{\{0,\dots,k\}} : x_0 = a, x_k = b, |x_i - x_{i-1}| = 1 \ \forall i = 1, \dots, n\}$. Show that $\#P_k(a, a + b) = \#P_k(a, a - b)$ and
$$\#P_k(1, -L) = \#\{x \in P_k(1, L) : 0 \in \{x_0, \dots, x_k\}\} \quad \text{for } L \ge 0 \text{ and } k \ge 1 .$$
Hint: Define a bijection between the two sets.
(iii) Let $A_k = \{X_{2i} \ne 0 \ \forall 1 \le i \le k\}$. Show that
$$\sum_{L=1}^{k} \#\{x \in P_{2k-1}(1, 2L) : 0 \notin \{x_1, \dots, x_{2k}\}\} = \#P_{2k-1}(1, 0)$$
and hence that $\mathbb{P}(A_k) = \mathbb{P}(X_{2k} = 0)$.
(iv) Establish that $\mathbb{P}(M_n = k) = \mathbb{P}(X_{2k} = 0)\mathbb{P}(A_{n-k})$ for $0 \le k \le n$.
(v) Show that for $0 < u < v < 1$ we have
$$\mathbb{P}(M_n/n \in (u, v)) \to \int_u^v \frac{dx}{\pi\sqrt{x(1 - x)}} \quad \text{as } n \to \infty .$$
(vi) Carefully deduce the main claim.

Exercise 3. Let (X_n) be a simple symmetric random walk on $\mathbb{Z}^d$ started at 0 for $d \geq 1$. We write $X_n = (X_n^1, \dots, X_n^d)$.

(i) Show that

$$\mathbb{E}((X_n^1)^2 + \cdots + (X_n^d)^2) = n \; .$$

Hint: Show that $X_n^i =_d \sum_{k=1}^n \sigma_k B_k$ where the (σ_k) are i.i.d. uniform on $\{-1, 1\}$ and (B_k) is an independent sequence of i.i.d. $B(1, 1/d)$ random variables.

(ii) Show that almost surely, for infinitely many $n \geq 1$ there exists $m > n$ such that $X_m \in \{X_0, \dots, X_n\}$.

Submission of solutions. Hand in by 20/12/2021 5 p.m. (online) following the instructions on the course website

`https://metaphor.ethz.ch/x/2021/hs/401-3601-00L/`

The exercise classes are listed below; the Zoom meeting details are given on the course website shown above.

Time	**Room**	**Assistant**
Tuesday 2 p.m. – 3 p.m.	HG F 26.5	Matthis Lehmkuehler
Tuesday 2 p.m. – 3 p.m.	ML H 41.1	Luca Pelizzari
Tuesday 3 p.m. – 4 p.m.	Zoom	Daniel Contreras Salinas
Tuesday 3 p.m. – 4 p.m.	ML H 41.1	Genc Kqiku