Prof. Wendelin Werner HS 2021

PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 2

Exercise 1. Consider a sequence of events (A_n).

(i) Show that the (A_n) are independent if and only if the sequence of σ-algebras $(\mathcal{F}_n)$ is independent where $\mathcal{F}_n = \{\emptyset, A_n, A_n^c, \Omega\}$.

(ii) Now suppose that the (A_n) are independent. Show that $\mathbb{P}(\cap_{n\geq 1} A_n) = \prod_{n\geq 1} \mathbb{P}(A_n)$.

Exercise 2. Let $(\mathcal{F}_n)$ be a sequence of independent σ-algebras and consider a bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$. Show that $(\mathcal{F}_{\sigma(n)})$ is still a sequence of independent σ-algebras.

Exercise 3. Fix $\alpha > 0$, $a \in \{0,1\}^k$ and let $k_* = a_1 + \cdots + a_k$. Now consider a sequence of independent events (A_n) with $\mathbb{P}(A_n) = 1/n^\alpha$ for all $n \in \mathbb{N}$ and let

$$N = \#\{n \in \mathbb{N} \colon (1_{A_n}, 1_{A_{n+1}}, \ldots, 1_{A_{n+k-1}}) = a\} .$$

If $\alpha k_* > 1$ show that $N < \infty$ almost surely. If $\alpha k_* \leq 1$ show that $N = \infty$ almost surely.

Exercise 4. Let X and Y be random variables taking values in a measure spaces $(E, \mathcal{E})$ and in $\mathbb{R}$ respectively. Also assume that Y is measurable from $(\Omega, \sigma(X))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We will show that there exists a measurable function $f\colon E \to \mathbb{R}$ such that $Y = f(X)$.

(i) Show that for all $A \in \sigma(X)$ there exists $g\colon E \to \mathbb{R}$ measurable such that $1_A = g(X)$.

(ii) Use the previous part, to show that for each $n \geq 1$ there are $f_n^\pm \colon E \to \mathbb{R}$ measurable such that

$$2^{-n}\lfloor (2^n Y) \vee 0)\rfloor = f_n^+(X) \quad \text{and} \quad 2^{-n}\lfloor (-2^n Y) \vee 0)\rfloor = f_n^-(X) .$$

(iii) Using that $f_n^+(X) - f_n^-(X) \to Y$ pointwise as $n \to \infty$, conclude the proof of the question.

Exercise 5*. Consider the probability space $(\Omega, \mathcal{B}([0,1)), \mathbb{P})$ where $\mathbb{P}$ denotes the Lebesgue measure on $[0,1)$. Recall also the definition of the symmetric difference

$$A \Delta B := (A \setminus B) \cup (B \setminus A) .$$

Show that for all $\epsilon > 0$ and $A \in \mathcal{B}([0,1))$ there exists a set $B = [a_1, b_1) \cup \cdots \cup [a_n, b_n)$ with $n \geq 1$ and $0 \leq a_1 < b_1 < \cdots < a_n < b_n \leq 1$ such that $\mathbb{P}(A \Delta B) < \epsilon$. Hint: Define a suitable Dynkin system and use Dynkin's lemma.

Submission of solutions. Hand in by 04/10/2021 5 p.m. (online) following the instructions on the course website

`https://metaphor.ethz.ch/x/2021/hs/401-3601-00L/`

The exercise classes are listed below; the Zoom meeting details are given on the course website shown above.

Time	Room	Assistant
Tuesday 2 p.m. – 3 p.m.	HG F 26.5	Matthis Lehmkuehler
Tuesday 2 p.m. – 3 p.m.	ML H 41.1	Luca Pelizzari
Tuesday 3 p.m. – 4 p.m.	Zoom	Daniel Contreras Salinas
Tuesday 3 p.m. – 4 p.m.	ML H 41.1	Genc Kqiku