PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 3

Exercise 1. Consider a sequence (X_n) of random variables and let $\sigma(X_n \colon n \geq 1)$ be the smallest σ-algebra containing $\sigma(X_m)$ for all $m \geq 1$. Show that

$$\mathcal{A} = \{\{X_1 \leq a_1, \ldots, X_m \leq a_m\} \colon m \geq 1\,,\ a \in \mathbb{R}^m\} \cup \{\Omega\}$$

is a π-system generating $\sigma(X_n \colon n \geq 1)$.

Exercise 2. Let (X_n) be a sequence of i.i.d. $N(0,1)$ random variables. Show that

$$\limsup_{n\to\infty} X_n/\sqrt{2\log n} = 1 \quad \text{almost surely.}$$

Hint: Use the Borel-Cantelli lemmas.

Exercise 3. Let U be uniformly distributed on $[0,1)$ and fix an integer $b \geq 2$. For each $n \geq 1$ we let

$$\epsilon_n = \lfloor b^n U \rfloor \mod b \in \{0, \ldots, b-1\}$$

i.e., ϵ_n is the nth digit in an expansion of U in base b. Show that (ϵ_n) is a sequence of i.i.d. random variables which are uniformly distributed on $\{0, \ldots, b-1\}$ and show that $U = \sum_{n\geq 1} \epsilon_n b^{-n}$ almost surely.

Exercise 4. Let X be a random variable taking values in $\mathbb{N}_0$ and let (X_n) be a sequence of i.i.d. random variables with the same law as X.

(i) Show that $\mathbb{E}(X) = \sum_{n\geq 1} \mathbb{P}(X \geq n)$.

(ii) Let us now make the assumption that $\mathbb{E}(X) = \infty$. Show that $\limsup_{n\to\infty} X_n/n \geq k$ almost surely for all $k \in \mathbb{N}$. Hint: First show that

$$\sum_{n\geq 1} \mathbb{P}(X \geq nk) = \infty\ .$$

(iii) Deduce that $\limsup_{n\to\infty} X_n/n = \infty$ almost surely.

Now consider any random variable Y satisfying $\mathbb{E}(|Y|) = \infty$ and let (Y_n) be i.i.d. random variables, each of which has the same law as Y.

(iv) Using the previous parts, show that $\limsup_{n\to\infty} |Y_n|/n = \infty$ almost surely.

(v) Deduce that $\limsup_{n\to\infty} |Y_1 + \cdots + Y_n|/n = \infty$ almost surely.

Exercise 5*. Let $(E_n, \mathcal{E}_n, P_n)$ be a probability spaces for each $n \geq 1$. Recall that for each $N \geq 1$ one can define a probability measure $P_1 \otimes \cdots \otimes P_N$ on $E_1 \times \cdots \times E_N$ endowed with the product σ-algebra $\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_N$ by

$$P_1 \otimes \cdots \otimes P_N(A) = \int_{E_1} P_1(dx_1) \cdots \int_{E_N} P_N(dx_N)\, 1((x_1, \ldots, x_N) \in A)$$

for $A \in \mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_N$. The goal of this question will be to perform this construction in the case of an infinite product. We define $E = \prod_{N\geq 1} E_N$ and let $\pi_N \colon E \to E_N$ be the projection map onto the Nth coordinate.

We also let $\mathcal{E} = \sigma(\pi_N^{-1}(A)\colon N \geq 1\,,\ A \in \mathcal{E}_N)$ be the product σ-algebra on E. Our goal will be to construct a probability measure P on $(E, \mathcal{E})$ with the property that $(\pi_1, \dots, \pi_N)_* P = P_1 \otimes \cdots \otimes P_N$ for all $N \geq 1$ where the left side is the pushforward of P along the map $(\pi_1, \dots, \pi_N)$.

(i) Let

$$\mathcal{A} = \left\{ B \times \prod_{N' > N} E_{N'} \ : N \geq 1\,,\ B \in \mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_N \right\} .$$

Show that this is an algebra, i.e. that it contains the empty set and that it is stable under taking complements and taking finite unions. Also show that $\mathcal{E} = \sigma(\mathcal{A})$.

(ii) For each set $A = B \times \prod_{N' > N} E_{N'} \in \mathcal{A}$ we define $P(A) = P_1 \otimes \cdots \otimes P_N(B)$. Show that P is well-defined.

Suppose that $(A_n) \subset \mathcal{A}$ is a non-increasing sequence of sets such that $P(A_n) \geq \epsilon$ for all $n \geq 1$ and a $\epsilon > 0$. We will show that $\cap_{n \geq 1} A_n \neq \emptyset$. To this end, for $N \geq 1$ we define

$$A_n^{(N)}(x_1, \dots, x_N) = \{(y_k) \in E\colon (x_1, \dots, x_N, y_{N+1}, \dots) \in A_n\}$$
$$B_n^{(N)}(x_1, \dots, x_{N-1}) = \{\tilde{x}_N \in E_N\colon P(A_n^{(N)}(x_1, \dots, x_{N-1}, \tilde{x}_N)) \geq \epsilon/2^N\}$$

with the convention that $A_n^{(0)} = A_n$ and where $(x_1, \dots, x_N, y_{N+1}, \dots)$ denotes the concatenation of the sequences $(x_1, \dots, x_N)$ and $(y_{N+k})_k$.

(iii) Show that for all $N \geq 0$, $(x_1, \dots, x_N) \in E_1 \times \cdots \times E_N$ and all $n \geq 1$, the following equality holds (and that both sides are well-defined)

$$P(A_n^{(N)}(x_1, \dots, x_N)) = \int_{E_{N+1}} P_{N+1}(d\tilde{x}_{N+1}) P(A_n^{(N+1)}(x_1, \dots, x_N, \tilde{x}_{N+1})) .$$

(iv) Hence establish that for $N \geq 0$,

$$P(A_n^{(N)}(x_1, \dots, x_N)) - \epsilon/2^{N+1} \leq P_{N+1}(B_n^{(N+1)}(x_1, \dots, x_N)) .$$

(v) We will now inductively construct a sequence $(x_N) \in E$ with the property that for each $N \geq 1$,

$$\epsilon/2^N \leq P_N(B_n^{(N)}(x_1, \dots, x_{N-1})) \quad \text{for all } n \geq 1 .$$

Hint: Having constructed $(x_1, \dots, x_{N-1})$, explain first why one can take

$$x_N \in \cap_{n \geq 1} B_n^{(N)}(x_1, \dots, x_{N-1})$$

and then use part (iv).

(vi) Show that $(x_N) \in A_n$ for all $n \geq 1$. Hint: Use that $A_n \in \mathcal{A}$.

The result derived above can also be reformulated to say that $P(A_n) \to 0$ as $n \to \infty$ whenever $(A_n) \subset \mathcal{A}$ is a non-increasing sequence of sets such that $\cap_{n\geq 1} A_n = \emptyset$.

(vii) Deduce that if $(A'_n) \subset \mathcal{A}$ is a sequence of disjoint sets with $A' := \cup_{n\geq 1} A'_n \in \mathcal{A}$ then

$$P(A') = \sum_{n\geq 1} P(A'_n)$$

i.e. countable additivity of P on $\mathcal{A}$ holds. Crucially, Carathéodory's extension theorem now implies that P extends to a measure on $\sigma(\mathcal{A})$.

(viii) Explain why P is a probability measure and why it satisfies the property $(\pi_1, \ldots, \pi_N)_* P = P_1 \otimes \cdots \otimes P_N$ for all $N \geq 1$.

Submission of solutions. Hand in by 11/10/2021 5 p.m. (online) following the instructions on the course website

`https://metaphor.ethz.ch/x/2021/hs/401-3601-00L/`

The exercise classes are listed below; the Zoom meeting details are given on the course website shown above.

Time	**Room**	**Assistant**
Tuesday 2 p.m. – 3 p.m.	HG F 26.5	Matthis Lehmkuehler
Tuesday 2 p.m. – 3 p.m.	ML H 41.1	Luca Pelizzari
Tuesday 3 p.m. – 4 p.m.	Zoom	Daniel Contreras Salinas
Tuesday 3 p.m. – 4 p.m.	ML H 41.1	Genc Kqiku