PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 4

Exercise 1. Let (X_n) be a sequence of independent random variables and let (a_n) be a deterministic sequence with $a_n \to 0$ as $n \to \infty$. Show using Kolmogorov's zero-one law that there exists (deterministic) constants $C_\pm \in [-\infty, \infty]$ such that

$$\liminf_{n\to\infty} a_n \cdot (X_1 + \cdots + X_n) = C_- \quad \text{and} \quad \limsup_{n\to\infty} a_n \cdot (X_1 + \cdots + X_n) = C_+$$

almost surely.

Exercise 2. The goal of this question is to walk through Etemadi's proof of the strong law of large numbers under a pairwise independence assumption. Let (X_n) be a sequence of pairwise independent, identically distributed and integrable random variables with mean μ. We will show that $(X_1 + \cdots + X_n)/n \to \mu$ as $n \to \infty$ almost surely.

(i) Show that it suffices to prove the result in the case where $X_n \geq 0$ for all $n \geq 0$. In the rest of the exercise, we will restrict to this scenario.

(ii) Let X have the same law as each of the (X_n). Show that for $\alpha > 1$ and $\epsilon > 0$

$$\sum_{m\geq 1} \mathbb{P}\left(\left| \frac{1}{\lfloor \alpha^m \rfloor} \sum_{i=1}^{\lfloor \alpha^m \rfloor} (X_i 1(X_i \leq i) - \mathbb{E}(X_i 1(X_i \leq i))) \right| > \epsilon \right)$$
$$\leq \epsilon^{-2} \sum_{m\geq 1} \frac{1}{\lfloor \alpha^m \rfloor^2} \sum_{i=1}^{\lfloor \alpha^m \rfloor} \mathbb{E}(X^2 1(X \leq i)) < \infty\,.$$

Hint: To see the finiteness it might prove helpful to interchange the sums.

(iii) Deduce that for all $\alpha > 1$,

$$\frac{1}{\lfloor \alpha^m \rfloor} \sum_{i=1}^{\lfloor \alpha^m \rfloor} X_i 1(X_i \leq i) \to \mathbb{E}(X) \quad \text{as } m \to \infty \text{ almost surely}\,.$$

(iv) Show that almost surely, $X_i \neq X_i 1(X_i \leq i)$ for at most finitely many indices i and then deduce that $(X_1 + \cdots + X_{\lfloor \alpha^m \rfloor})/\lfloor \alpha^m \rfloor \to \mathbb{E}(X)$ as $m \to \infty$ almost surely.

(v) Using that $(X_n) \geq 0$ obtain that for all $\alpha > 1$,

$$\mathbb{E}(X)/\alpha \leq \liminf_{n\to\infty} (X_1 + \cdots + X_n)/n$$
$$\leq \limsup_{n\to\infty} (X_1 + \cdots + X_n)/n \leq \alpha\, \mathbb{E}(X) \quad \text{almost surely}\,.$$

(vi) Carefully deduce the strong law of large numbers from (v).

Exercise 3. Let (X_n) be an i.i.d. sequence with the same law as X such that $\mathbb{E}(X^{2p}) < \infty$ for all integers $p \geq 1$. Also assume that $\mathbb{E}(X) = 0$.

(i) Show that for all integers $p \geq 1$ there exists a constant $C_p < \infty$ such that

$$\mathbb{E}\left((X_1 + \cdots + X_n)^{2p}\right) \leq C_p\, n^p \,.$$

(ii) Deduce that $(X_1 + \cdots + X_n)/n^{1/2+\delta} \to 0$ as $n \to \infty$ almost surely for all $\delta > 0$.

Submission of solutions. Hand in by 18/10/2021 5 p.m. (online) following the instructions on the course website

`https://metaphor.ethz.ch/x/2021/hs/401-3601-00L/`

The exercise classes are listed below; the Zoom meeting details are given on the course website shown above.

Time	**Room**	**Assistant**
Tuesday 2 p.m. – 3 p.m.	HG F 26.5	Matthis Lehmkuehler
Tuesday 2 p.m. – 3 p.m.	ML H 41.1	Luca Pelizzari
Tuesday 3 p.m. – 4 p.m.	Zoom	Daniel Contreras Salinas
Tuesday 3 p.m. – 4 p.m.	ML H 41.1	Genc Kqiku