

**PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 5**

Exercise 1. Suppose that (X_n) are i.i.d. random variables taking values in $(0, \infty)$ with the same law as X . Also suppose that $\mathbb{E}(|\log X|) < \infty$.

(i) Show that

$$X_1 \cdots X_n = e^{\alpha n + o(n)} \quad \text{as } n \rightarrow \infty \text{ almost surely}$$

where $\alpha = \mathbb{E}(\log X)$. Hint: First rewrite the statement without the $o(n)$ notation.

(ii) Fix $a > 1$. Construct a sequence (Y_n) with values in $(0, \infty)$ of random variables such that $\mathbb{E}(Y_n) = a^n$ for all $n \geq 1$ and $Y_n \rightarrow 0$ almost surely as $n \rightarrow \infty$.

Exercise 2. Suppose that (U_n) are i.i.d. uniform random variables on $\{\pm 1\}$ and consider $\alpha > 0$. Discuss the convergence of the series $\sum_{n \geq 1} U_n n^{-\alpha}$.

Exercise 3. This question is related to the three series criterion and its converse. We consider a sequence (X_n) of i.i.d. integrable random variables with the same law as X and $\mathbb{E}(X) = 0$.

- (i) Show that if $\mathbb{E}(X^2) < \infty$ then $\sum_{n \geq 1} X_n/n$ converges almost surely.
- (ii) Suppose now instead that X and $-X$ have the same law. Show that in this case the series $\sum_{n \geq 1} X_n/n$ converges almost surely as well.
- (iii) Find a sequence (X_n) for which the series $\sum_{n \geq 1} X_n/n$ does not converge with positive probability.
- (iv) Deduce that with the sequence (X_n) as in (iii), the series $\sum_{n \geq 1} X_n/n$ does not converge almost surely.

Exercise 4. Let (X_n) be i.i.d. integrable random variables with the same law as X . Define

$$M_n = \frac{1}{n} \sum_{i=1}^n X_i X_{i+1}.$$

Show that the sequence (M_n) converges almost surely and find its (almost sure) limit.

Submission of solutions. Hand in by 25/10/2021 5 p.m. (online) following the instructions on the course website

<https://metaphor.ethz.ch/x/2021/hs/401-3601-00L/>

The exercise classes are listed below; the Zoom meeting details are given on the course website shown above.

Time	Room	Assistant
Tuesday 2 p.m. – 3 p.m.	HG F 26.5	Matthis Lehmkuehler
Tuesday 2 p.m. – 3 p.m.	ML H 41.1	Luca Pelizzari
Tuesday 3 p.m. – 4 p.m.	Zoom	Daniel Contreras Salinas
Tuesday 3 p.m. – 4 p.m.	ML H 41.1	Genc Kqiku