

**PROBABILITY THEORY (D-MATH)  
EXERCISE SHEET 6**

**Exercise 1.** Let  $X$  and  $Y$  be two independent Bernoulli distributed random variables with parameter  $p$ . Define  $Z = 1(X + Y = 0)$  and  $\mathcal{G} = \sigma(Z)$ . Find  $\mathbb{E}(X | \mathcal{G})$  and  $\mathbb{E}(Y | \mathcal{G})$ . Are these random variables also independent?

**Exercise 2.** Let  $X$  and  $Y$  be independent random variables taking values in measure spaces  $(E, \mathcal{E})$  and  $(E', \mathcal{E}')$  respectively. Suppose also that  $f: E \times E' \rightarrow \mathbb{R}$  is a bounded measurable function. Show that

$$\mathbb{E}(f(X, Y) | Y) = g(Y) \quad \text{a.s.}$$

where  $g(y) = \mathbb{E}(f(X, y))$ . In particular, explain why  $g: E' \rightarrow \mathbb{R}$  is measurable.

**Exercise 3.** Suppose that  $X, Y$  are integrable random variables and that

$$\mathbb{E}(Y | X) = X \quad \text{and} \quad \mathbb{E}(X | Y) = Y \quad \text{a.s.}$$

Prove that  $X = Y$  almost surely. Hint: Consider the expression

$$\mathbb{E}((X - Y)1(X > c, Y \leq c)) + \mathbb{E}((X - Y)1(X \leq c, Y \leq c)).$$

**Exercise 4.** Consider a  $\sigma$ -algebra  $\mathcal{G}$  contained in the  $\sigma$ -algebra of the underlying probability space. Moreover, consider non-negative random variables  $(X_n)$ . Show that

$$\mathbb{E}\left(\liminf_{n \rightarrow \infty} X_n | \mathcal{G}\right) \leq \liminf_{n \rightarrow \infty} \mathbb{E}(X_n | \mathcal{G}) \quad \text{a.s.}$$

This is Fatou's lemma for conditional expectations.

**Exercise 5.** Let  $E$  be an exponentially distributed random variable with parameter 1 and define  $\mathcal{G}_t = \sigma(E \wedge t, 1(E \leq t))$  for  $t \geq 0$ . Show that for any  $t \geq 0$  and any measurable function  $f: \mathbb{R} \rightarrow [0, \infty]$  we have

$$\mathbb{E}(f(E) | \mathcal{G}_t) = \mathbb{E}(f(E + t)1(E > t) + f(E)1(E \leq t)) \quad \text{a.s.}$$

**Submission of solutions.** Hand in by 01/11/2021 5 p.m. (online) following the instructions on the course website

<https://metaphor.ethz.ch/x/2021/hs/401-3601-00L/>

The exercise classes are listed below; the Zoom meeting details are given on the course website shown above.

Time	Room	Assistant
Tuesday 2 p.m. – 3 p.m.	HG F 26.5	Matthis Lehmkuehler
Tuesday 2 p.m. – 3 p.m.	ML H 41.1	Luca Pelizzari
Tuesday 3 p.m. – 4 p.m.	Zoom	Daniel Contreras Salinas
Tuesday 3 p.m. – 4 p.m.	ML H 41.1	Genc Kqiku