PROBABILITY THEORY (D-MATH)
EXERCISE SHEET 9

Exercise 1. Let $S, T : \Omega \to \mathbb{N} \cup \{\infty\}$ be $(\mathcal{F}_n)$ stopping times. Prove or provide a counter example disproving the following statements:
(i) $S - 1$ is a stopping time.
(ii) $S + 1$ is a stopping time.
(iii) $S \wedge T$ is a stopping time.
(iv) $S \vee T$ is a stopping time.
(v) $S + T$ is a stopping time.

Exercise 2. Let $(A_n : n \in \mathbb{N})$ be a sequence of independent events all having probability $p > 0$. Fix $N \in \mathbb{N}$ and write T for the first time we achieve a run of N consecutive successes. Thus

$$T = \inf\{n \geq N \colon 1_{A_n} + \cdots + 1_{A_{n-N+1}} = N\} .$$

Show that $\mathbb{P}(T \geq n) \leq C\alpha^n$ for some constants $C < \infty$ and $\alpha \in (0, 1)$, and hence that $\mathbb{E}(T) < \infty$.

Exercise 3. (Wald's identities) Let (X_n) be i.i.d. and $\mathbb{P}(X_1 = 0) < 1$. Now let $S_0 = 0$, $S_n = X_1 + \cdots + X_n$ for $n \geq 1$ and for $a \leq 0 \leq b$ we define $T = \inf\{n \geq 0 \colon S_n \notin (a, b)\}$. Finally, let us define a filtration $(\mathcal{F}_n)$ by $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.
(i) Suppose that X_1 is integrable with mean μ. Show that T is an integrable stopping time for $(\mathcal{F}_n)$ and establish that $\mathbb{E}(S_T) = \mu\mathbb{E}(T)$.
(ii) Suppose that X_1 is in L^2 with mean 0 and variance σ^2. Show that

$$\mathbb{E}\left(S_T^2\right) = \sigma^2\, \mathbb{E}(T) .$$

(iii) Suppose that $\lambda \in \mathbb{R}$ is such that $\mathbb{E}(e^{\lambda X_1}) = 1$. Show that

$$\mathbb{E}\left(\exp(\lambda S_T)\right) = 1 .$$

(iv) Suppose now that the law of X_1 is $p\delta_1 + (1 - p)\delta_{-1}$ and consider $a, b \in \mathbb{Z}$. Explicitly determine $\mathbb{E}(T)$ and $\mathbb{P}(S_T = b)$.

Exercise 4. (Azuma's inequality) Let (M_n) be a martingale starting from 0 with respect to a filtration $(\mathcal{F}_n)$ with $|M_n - M_{n-1}| \leq c_n$ for all $n \geq 1$ and finite deterministic constants $c_n < \infty$.

(i) Show that if Y is a random variable with mean 0 and $|Y| \leq c$ then for $\theta \in \mathbb{R}$,

$$\mathbb{E}(e^{\theta Y}) \leq \cosh(\theta c) \leq e^{\theta^2 c^2/2} .$$

Hint: Use the convexity of $y \mapsto e^{\theta y}$ on $[-c, c]$.

(ii) Show that for $\theta \in \mathbb{R}$,

$$\mathbb{E}(e^{\theta M_n}) \leq e^{\theta^2 \sigma_n^2/2}$$

where $\sigma_n^2 = c_1^2 + \cdots + c_n^2$.

(iii) Deduce that for $x \geq 0$,

$$\mathbb{P}\left(\sup_{k \leq n} M_k \geq x\right) \leq e^{-x^2/(2\sigma_n^2)} .$$

Submission of solutions. Hand in by 22/11/2021 5 p.m. (online) following the instructions on the course website

`https://metaphor.ethz.ch/x/2021/hs/401-3601-00L/`

The exercise classes are listed below; the Zoom meeting details are given on the course website shown above.

Time	Room	Assistant
Tuesday 2 p.m. – 3 p.m.	HG F 26.5	Matthis Lehmkuehler
Tuesday 2 p.m. – 3 p.m.	ML H 41.1	Luca Pelizzari
Tuesday 3 p.m. – 4 p.m.	Zoom	Daniel Contreras Salinas
Tuesday 3 p.m. – 4 p.m.	ML H 41.1	Genc Kqiku