Ising Model: Exercise sheet 1

We recall that exercise with * can be handover.

Exercise 1. Let $n \geq 1$, $d \geq 1$, $\beta \geq 0$. Set $\Lambda = \{-n, \dots, n\}^d$. We recall that $\overline{1}$ is the state where all the spins are equal to 1. Prove that

$$\forall \sigma \in \{0,1\}^{\Lambda} \qquad \lim_{\beta \to \infty} \mu_\beta(\sigma) = \begin{cases} \frac{1}{2} & \text{if } \sigma = \overline{1} \text{ or } \sigma = \overline{-1} \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 2. * Let Ω be a finite state space. To each state $\omega \in \Omega$ we associate its energy $H(\omega)$. Let μ be a measure on Ω. We define $S(\mu)$ the entropy of μ as follows

$$S(\mu) := -\sum_{\omega \in \Omega} \mu(\omega) \log \mu(\omega).$$

The aim of this exercise is to prove that under the constraint that the mean energy

$$h(\mu) := \sum_{\omega \in \Omega} \mu(\omega) H(\omega)$$

is equal to some $h_0 \in (\min_{\omega \in \Omega} H(\omega), \max_{\omega \in \Omega} H(\omega))$, the measure μ that maximizes the entropy is a Gibbs measure (that is of the form $\mu(\omega) = C \exp(-cH(\omega))$).

1. Prove that the function
$$f : (x_1, \dots, x_n) \mapsto \sum_{1 \leq i \leq n} x_i \log x_i$$
is differentiable on $(0,1]^n$ and convex on $[0,1]^n$.

2. Let us denote by $\nabla f(x)$ the gradient of the function f at x. Prove that
$$\forall x, y \in (0,1]^n \qquad f(y) - f(x) \geq \langle \nabla f(x), y - x \rangle$$
where $\langle x, y \rangle$ is the scalar product on $\mathbb{R}^n$ between x and y.

3. Let $(h_1, \dots, h_n) \in \mathbb{R}^n$. Let $h_0 \in (\min_{1 \leq i \leq n} h_i, \max_{1 \leq i \leq n} h_i)$. Set
$$E = \left\{ x \in (0,1]^n : \sum_{1 \leq i \leq n} x_i = 1, \sum_{1 \leq i \leq n} x_i h_i = h_0 \right\}.$$
Prove that E is not empty.

4. Prove that if there exists $x \in E$ such that $\nabla f(x)$ is colinear with $(h_1, \dots, h_n)$ then x is a minimizer of f on E.

5. Prove that if x is a minimizer on E then it is the unique minimizer on
$$\left\{ x \in [0,1]^n : \sum_{1 \leq i \leq n} x_i = 1, \sum_{1 \leq i \leq n} x_i h_i = h_0 \right\}.$$
Conclude by giving the expression of μ depending on h_0.