Ising Model: Exercise sheet 11

In this exercise sheet, we aim to study the Ising model with an external field. We will first extend some results of the lecture in that context. We aim to prove that in presence of an external field, for any temperature $\beta \geq 0$, there exists a unique infinite volume measure.

We fix $\beta \geq 0$. Let $\Lambda \subset\subset \mathbb{Z}^d$. The Ising measure in Λ with boundary conditions ω, inverse temperature β and external field $h \in \mathbb{R}$ is defined by

$$\forall \sigma \in \{-1,1\}^\Lambda \quad \mu^\omega_{\Lambda,h}(\sigma) = \frac{1}{Z^\omega_{\Lambda,h}} e^{-H^\omega_{\Lambda,h}(\sigma)}$$

where

$$H^\omega_{\Lambda,h}(\sigma) = -\beta \sum_{\substack{xy \subset \Lambda \\ x\sim y}} \sigma_x \sigma_y - \beta \sum_{\substack{x\in\Lambda,\, y\in\partial\Lambda \\ x\sim y}} \omega_y \sigma_x - h \sum_{x\in\Lambda} \sigma_x$$

and

$$Z^\omega_{\Lambda,h} = \sum_{\sigma\in\{-1,1\}^\Lambda} e^{-H^\omega_{\Lambda,h}(\sigma)}.$$

Exercise 1 (Definition of the magnetization and properties). We define the magnetization

$$\forall h \in \mathbb{R} \qquad m(h) := \langle \sigma_0 \rangle^+_h = \lim_{n\to\infty} \langle \sigma_0 \rangle^+_{\Lambda_n,h}\,.$$

1. Prove that if $\omega \leq \omega'$ the measure $\mu^\omega_{\Lambda,h}$ is stochastically dominated by $\mu^{\omega'}_{\Lambda,h}$.
2. Prove that $\mu^\omega_{\Lambda,h}$ satisfies the domain Markov property. Deduce that the sequence $(\langle \sigma_0 \rangle^+_{\Lambda_n,h})_{n\geq 1}$ is non-increasing. Conclude that the limit in $m(h)$ is well-defined.
3. Let $n \geq 1$. Prove that the function $h \mapsto \langle \sigma_0 \rangle^+_{\Lambda_n,h}$ is non-decreasing.
4. Conclude that the function $h \mapsto m(h)$ is non-decreasing and right-continuous on $\mathbb{R}$.
5. Prove similarly that $\langle \sigma_0 \rangle^-_h = \lim_{n\to\infty} \langle \sigma_0 \rangle^-_{\Lambda_n,h}$ is well-defined and that $h \mapsto \langle \sigma_0 \rangle^-_h$ is left-continuous and non-decreasing.

Exercise 2 (Characterization of the uniqueness of the infinite volume measure). In this exercise, we aim to give a characterization for the uniqueness of the infinite volume measure when there is an external field. We use the notation $\Delta_n \uparrow \mathbb{Z}^d$ for a sequence of subset of $\mathbb{Z}^d$ such that $\Delta_k \subset \Delta_{k+1}$ for any $k \geq 1$ and $\mathbb{Z}^d = \cup_{k\geq 1} \Delta_k$.

1. Prove that there exists two probability measures μ^+_h and μ^-_h characterized by respectively

$$\forall f \text{ local functions} \quad \forall \Delta_n \uparrow \mathbb{Z}^d \quad \langle f \rangle^+_h = \lim_{k\to\infty} \langle f \rangle^+_{\Delta_k,h}$$

and

$$\forall f \text{ local functions} \quad \forall \Delta_n \uparrow \mathbb{Z}^d \quad \langle f \rangle^-_h = \lim_{k\to\infty} \langle f \rangle^-_{\Delta_k,h}\,.$$

2. Prove that $\mu^+_h = \mu^-_h$ if and only if $\langle \sigma_0 \rangle^-_h = \langle \sigma_0 \rangle^+_h$. We assume in what follows that if $\mu^+_h = \mu^-_h$ then there is uniqueness of the infinite volume measure.

Exercise 3 (Uniqueness of the infinite volume measure). In this exercise we aim to prove that $h > 0$ there exists a unique infinite volume measure.

1. Prove that
$$\lim_{\substack{h'\to h\\ h'<h}} \langle \sigma_0\rangle^+_{h'} \geq \langle \sigma_0\rangle^-_h \,.$$

2. Set $S = \sum_{x\in\Lambda_n} \sigma_x$ and $S' = \sum_{x\in\Lambda_n, y\in\partial\Lambda_n, x\sim y} \sigma_x$. Let $f : \Omega_{\Lambda_n} \to \mathbb{R}$ prove that
$$\langle f\rangle^+_{\Lambda_n,h'} = \frac{\left\langle f e^{(h'-h)S+2\beta S'}\right\rangle^-_{\Lambda_n,h}}{\left\langle e^{(h'-h)S+2\beta S'}\right\rangle^-_{\Lambda_n,h}}. \tag{1}$$

3. Let $h' < h$. Set $a := \langle\sigma_0\rangle^+_{h'}$ and $b := \langle\sigma_0\rangle^-_h$. Prove that
$$\langle S\rangle^+_{\Lambda_n,h'} \geq a\,|\Lambda_n|$$
and
$$\langle S\rangle^-_{\Lambda_n,h} \leq b\,|\Lambda_n|$$
.

4. Let $\epsilon > 0$. Prove using Markov inequality that
$$\mu^+_{\Lambda_n,h'}\left[S \leq (a-\epsilon)|\Lambda_n|\right] \leq 1 - \frac{\epsilon}{2}$$
and
$$\mu^-_{\Lambda_n,h}\left[S \leq (b+\epsilon)|\Lambda_n|\right] \leq 1 - \frac{\epsilon}{2}.$$

5. Using $f = \mathbf{1}_{S\geq(a-\epsilon)|\Lambda_n|}$ in (1), prove that
$$\frac{\epsilon}{2} \leq \frac{e^{(h'-h)(a-\epsilon)|\Lambda_n|+2\beta|\partial\Lambda_n|}}{\left\langle e^{(h'-h)S+2\beta S'}\right\rangle^-_{\Lambda_n,h}}$$
and
$$\left\langle e^{(h'-h)S+2\beta S'}\right\rangle^-_{\Lambda_n,h} \geq \frac{\epsilon}{2} e^{(h'-h)(b+\epsilon)|\Lambda_n|-2\beta|\partial\Lambda_n|}\,.$$

6. Using the previous question prove that $a - b - 2\epsilon \leq 0$. Deduce that
$$\lim_{\substack{h'\to h\\ h'<h}} \langle \sigma_0\rangle^+_{h'} \leq \langle \sigma_0\rangle^-_h \,.$$

7. Deduce from the previous questions that there exists a unique infinite volume measure at h if and only if the function m is continuous at h.

8. Using GHS inequality (see the exercise below) prove that the function $h \mapsto \langle\sigma_0\rangle^+_{\Lambda_n,h}$ is concave on $\mathbb{R}_+$. Deduce the continuity of m and that for $h > 0$ there exists a unique infinite volume measure.

Exercise 4 (GHS inequality). Let $\Lambda \subset \mathbb{Z}^d$. Let $x, y, z \in \Lambda$ and $h \geq 0$. Prove that
$$\langle \sigma_x;\sigma_y;\sigma_z\rangle^+_{\Lambda,h} \leq 0$$
where
$$\langle \sigma_x;\sigma_y;\sigma_z\rangle := \langle\sigma_x\sigma_y\sigma_z\rangle - \langle\sigma_x\rangle\langle\sigma_y\sigma_z\rangle - \langle\sigma_y\rangle\langle\sigma_x\sigma_z\rangle - \langle\sigma_z\rangle\langle\sigma_x\sigma_y\rangle + 2\langle\sigma_x\rangle\langle\sigma_y\rangle\langle\sigma_z\rangle\,.$$

Hint: First prove using the random current representation that for a finite graph $G = (V, E)$ with coupling constants J we have for $x, y, z, t \in V$
$$\langle\sigma_t\sigma_x\sigma_y\sigma_z\rangle \leq \langle\sigma_t\sigma_x\rangle\langle\sigma_y\sigma_z\rangle + \langle\sigma_t\sigma_y\rangle\langle\sigma_x\sigma_z\rangle + \langle\sigma_t\sigma_z\rangle\langle\sigma_x\sigma_y\rangle - 2\langle\sigma_t\sigma_x\rangle\langle\sigma_t\sigma_y\rangle\langle\sigma_t\sigma_z\rangle\,.$$